AN INVESTIGATION INTO PROBLEM SOLVING SKILLS IN CALCULUS: THE CASE OF UNISA FIRST YEAR STUDENTS

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by

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Summary

Students' performances in mathematics in an Open Distant Learning setting have not always been impressive. An exploratory study into the problem solving skills of the University of South Africa students in the Calculus module MAT112 is being conducted using past examinations scripts between 2006 and 2009. The study re-assesses the work done in the end-of-year Calculus examinations, by both looking at the distribution of marks awarded and assigning new scores based on an assessment rubric adapted for the problem at hand. Further assessment of qualitative dimensions that is important for problem solving in Calculus is developed from the data obtained from the assessment rubric. Using factor analysis, a hesitation factor, transfer-of-knowledge factor as well as ingenuity factor, are identified in successful Calculus problem solving. The study proposes two conceptual models; the first is to guide students in solving Calculus problems while the second one is meant to assist lecturers in the assessment of students of Calculus.

Keywords

Value of Calculus, problem solving skills, Factor analysis, hesitation factor, transfer-of-knowledge, essence of Mathematics, qualitative assessment, problem of poor performances, rigour and proof in mathematics, problem solving approach in Mathematics, Calculus assessment with the rubric, mathematics in an ODL environment.

DECLARATION

Student number: 30-445-485

I declare that

An Investigation into Problem Solving Skills in Calculus: The Case of UNISA First Year Students

is my own work and that all sources that I have used or quoted have been indicated and acknowledged by means of complete references.

………………………………… ……………………………

Signature Date

(MRS STELLA MUGISHA)

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I wish to extend my sincere gratitude to my supervisor, Professor L.D. Mogari, who guided me through all the stages of this work. He read my earlier versions of the dissertation and made very valuable comments, which assisted me in improving the focus and narrowing down the scope of the study.

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I would like to acknowledge the encouragement I received from my colleagues in the Department of Mathematics and outside at UNISA. The very keen students of Calculus whom I have taught over the years were the very reason why I delved into the area of student performance, so that I might be able to help all Calculus students to perform better. I am indebted to Dr. Segun Adeyefa for the useful discussions he held with me.

I spent many long hours away from home doing the research for this dissertation. My family missed me at home, and I wish to thank them for their patience and encouragement.

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CHAPTER I: INTRODUCION OF THE STUDY

1.1 Background of the Study

Mathematics is a necessity that was borne out of problems created by human interactions, from the time humans saw the need to quantify their possessions and compare their wealth with others, to the time of performing great scientific and technological feats (Adeyefa, 2012a).

Mathematics is evolving and becoming more complicated as human endeavours and interactions are becoming more complex. It enhances development and the achievement of sophistication in Science and Technology; the construction of high rise building, the manufacturing of supercomputers, exotic cars, the landing of spacecrafts on the moon and the planet mars, the launching of satellites, the use of internet banking, to mention but a few; hence the slogan Mathematics is the Queen of Science and the mother of Technology (Adeyefa, 2012b).

The role of Mathematics in Commerce and Industry is recognized worldwide as a necessity for advancing Science and Technology. According to Marks, Purdy and Kinney (1970 p. 4):

> "*Business and Industry suffer from a shortage of personnel who understand basic mathematical ideas, such as recognizing the mathematical relationships in an unfamiliar situation and applying mathematical reasoning to develop tentative conclusions and test their validity*"

Calculus is the branch of mathematics that deals with derivatives, infinite series, integrals and limits. The role of calculus in economic analysis cannot be over emphasized. For example policy economists analyse relationships between variables in order to understand, predict, plan for, and influence behaviour, among other things. In order to understand the sophisticated, complex behaviour of economic agents in the marketplace, then, we have to be able to model that behaviour using complex, nonlinear functions and then analyse our models appropriately. More simply, real world behaviour is not linear. Therefore, any attempt to understand real economic behaviour requires nonlinear modelling and the appropriate use of calculus. Any other, simpler forms of analysis lack the detail and sophistication essential to creating and evaluating policy.

Worldwide, the study of Mathematics is credited with the potential to solve everyday problems. This is because according to Davis (1984), problem solving in Mathematics gives one the ability: to learn and apply rules, to understand that a problem can be solved in more than one way; to learn to see relationships among different entities; and to learn to know that specific operations are needed in solving a problem.

Michaels and Smit (2008) argue in favour of horizontal integration of Mathematics and Science with a view to enhancing learners' potential. They recommend problem-based learning approach. Their view in this regard is that the problem-based learning approach is a means to helping learners realise the connections in different subjects, seeing relationships within what they learn and perceive their field of study in a broader perspective. The authors argue further that the approach improves learners' comprehension, synthesis and translation that lead to an integrated knowledge base, promotion of understanding, enhancement of problem solving abilities and self directed learning.

At the personal level, Mathematics is an intellectual exercise that challenges the mind. It gives immense pleasure and satisfaction when one correctly solves a mathematical problem. According to Maslow's hierarchy of needs, a person's activity is directed at satisfying the basic needs of clothing, food and shelter (at the bottom) and of satisfying the esoteric needs of self-actualization at the top (Gilbert, 2011). The top-most needs are those of an intellectual nature such as being able to successfully provide a solution to a mathematical problem. It is through such feat that humans attain self-actualization.

Mathematics is also based on the application of rules to mathematical objects such as in numbers and geometric figures (Ross and Rakow, 1982). A mathematical problem cannot be solved correctly unless one understands the rules and applies them correctly; and in order to apply such rules, one requires a certain amount of discipline. The importance of discipline is soon carried over to all grades and subsequently to the study of Mathematics at the university level.

Obeying rules must be accompanied by correct application of those rules. It is the ability to apply rules to different situations that forms the basis for successful mathematical problem solving (HodnikČade and Škrbe 2011). Therefore, when children in elementary school are taught Mathematics, they immediately learn that, it is good to listen and pay attention, because if one does not observe rules, one is not likely to determine correct answers to mathematical problems. From the early days of schooling, children are taught different ways of finding answers to mathematical problems; because it encourages independent thinking (HodnikČade and Škrbe 2011). University students, on the other hand, are taught to consider the context of problems such that they are able to see relationships between elements; so as to comprehend "what is given" and "what is wanted" in order to move on to the next step of solving a mathematical problem correctly. Being able to see and understand relationships is a key to successful problem solving in Mathematics (Bunker, 1969 p. 154).

In discussing the nature of Mathematics, Stuart and Ted (2011) argue that Mathematics, although fallible, is a practice that produces objective problem solving situations that are independent of individual cognition or social acceptance. They further argue that Mathematics will either be reduced to the experience of the individual and his or her personal knowledge, or to problem solving that is related in its entirety to the narrow aspirations and social situation of the class.

Ernest (1991, p.42) observes the following:

i) objectivity itself will be understood to be social; and
ii) publication or outward expression is necessary but not sufficient for subjective knowledge to become objective mathematical knowledge.

In accordance with Stuart and Ted (2011) it is mathematical practice that determines the objectivity of Mathematics, yet its objectivity consists precisely having validity independently of individual cognition or social acceptance.

The absolutist view of Mathematics is as a body of knowledge whose

> *"truths appear to everyone to be necessary and certain"* (Ayer, 1991, p. 7).

The whole of this rests on certain assumptions, which are held to be self-evident. Indeed, strict absolutist holds that Mathematics is almost independent of humankind, existing as it does in its government of nature, binding the universe together with its unfailing consistency across time and space. This consistency has been, to the absolutists, one of Mathematics' most powerful appeals (Hall, 2011).

Students at the Universities choose Mathematics because it is a prerequisite for the fields of study; such as Natural Science, Economics, Engineering, Medicine, Management as well as Accountancy. Mathematics as a learning area has composite topics that have varying uses, both in academy and real life. Calculus, for example, is a body of knowledge that is widely used in Physics, Chemistry, Engineering, Economics, Statistics, Computer Science, and so on.

The study of Calculus has inherent value in that it requires all of the student's prior mathematical knowledge of Algebra, Trigonometry and Coordinate Geometry, thus helping the Calculus student to begin to see Mathematics as a whole rather than just as individual segments. Furthermore, Calculus students continuously make use of problem solving skills, which are the essential element for many study areas, particularly those with problems involving maximizing revenues and minimizing costs. Other disciplines that utilize Calculus include those, where growth of populations or colonies is important, and where means and variances are needed to describe the behaviour of statistical populations (Texas Education Agency, 1990).

Apart from these advantages there are many other benefits derivable from learning Calculus. These include (Naidoo, 1985, p.16): development of intellectual potential of learners in higher education institutions by helping students to organize and interpret given information and to reach correct answers and conclusions by accurate and logical reasoning as well as;

i) Learning the correct use of symbols and concepts
ii) Understanding a variety of concepts and seeing important relationships among them
iii) Using everyday examples to see how Mathematics can be used profitably in solving common problems, and
iv) Developing the ability to communicate in mathematical language which is a language of commerce and industry.

Indeed, learning Mathematics in general teaches a learner to adopt a systematic approach to all subjects in all grades; from school level to university (Naidoo, 1985, p.18).

At the University of South Africa, the students' performances in the Calculus course MAT112 have not always been impressive over the years. The analysis of the results between 2006 and 2009 is given as follows: the number of students who wrote in Oct/Nov 2006, 2007, 2008, and May/June2009 were 926, 827, 1181 and 539 respectively; and the corresponding number of candidates who passed were 351, 326, 582 and 212 respectively (UNISA Exams, 2006, 2007, 2008 2009).

Over this period the highest pass rate (49.2%) was recorded in Oct/Nov 2008. The year Oct/Nov 2006 witnessed the worst performances of students, with 37.9% pass rate. Oct/Nov 2007 and May/Jun 2009 had 39.4% and 39.3% pass rate respectively.

This development has called for an investigation into problem solving skills of first year students in Calculus, using UNISA as a case study. This is due to the fact that UNISA is one of the largest universities in the world operating in Open and Distance Learning (ODL) environment.

1.2 The research problem

The importance of Mathematics in general and Calculus in particular cannot be over-emphasized, and the poor performances of MAT112 students, over the years, are of great concern to the Department of Mathematical Sciences, and the University in general.

This research shall attempt to answer the following questions:

i) Do the students understand the link between limits and differentiation?

ii) To what extent are problems on limit difficult compared with those on differentiation?

ii) To what extent are problems on limits and differentiation distributed on the basis of their level of difficulty

iii) To what extent do students perform in problems on integration?

iv) To what extent do students perform in problems on differential equations?

v) How effective is the Analytical Scale for Problem Solving rubric?

vi) To what extent do students transfer knowledge from limits to differentiation, from differentiation to integration, from differentiation to differential equations, and from integration to differential equations?

vii) What are the implications of using factor analysis on rubric scores?

1.3 Aim

In this dissertation the researcher is set to investigate how the students of MAT112 have been answering examination questions with a view to understanding their problems in mastering the subject.

1.4 Objective of the study

The main objective of the study is to analyse the following:

i) The level of problem solving skills displayed by students in answering questions in a first year Calculus examination at the University of South Africa (UNISA);

ii) The type of difficulties students of Calculus are faced with in answering questions;

iii) Investigate alternative ways of assessing the performance of students in a Calculus examination with a view to designing two conceptual models that will improve the students' problem solving skills and alleviate problems encountered on one hand, and help Mathematics teachers in marking Calculus examinations on the other.

1.5 Purpose of the study

The purpose of the study is two-fold; the first is related to the lecturer while the second concerns the students; and they are as follows:

i) To help the lecturer assess the students' ability.

The study shall examine the following variables, among others:

a) The formal application of rules;
b) Recognition of a solution strategy or right approach;
c) Skill in solving a problem to its logical conclusion;
d) Hesitation or lack of it in solving a Calculus problem;
e) Honesty in answering a question; and
f) Ingenuity in overcoming obstacles during the process of problem solving.

ii) The study will help the students to achieve the following:

a) Organize and interpret given information to reach correct answers and conclusions by accurate and logical reasoning;
b) Learn the correct use of symbols and concepts;
c) Understand a variety of concepts and seeing important relationships among them;
d) Use everyday examples to see how Mathematics can be used profitably in solving common problems; and
e) Develop the ability to communicate in mathematical language.

1.6 Significance of the study

Already within the South African education systems there are opportunities for the introduction of outcome-based education (OBE) curricula at tertiary level, yet the way the learners in OBE are assessed is likely to present immediate challenges to current lecturers of Mathematics at the Universities. Whereas OBE emphasizes the synthesis of mathematical knowledge with immediate practice, the traditional mode of assessment of Mathematics students at tertiary level has emphasized solving a large number of problems in order to impart mathematical skills irrespective of whether the problems solved have immediate practical relevance or not. Rigour and proof are other areas that have traditionally been considered important at university level.

The University lecturers should not be caught off guard; that is to say, they should not be forced to participate in a new OBE paradigm of learning and assessment without first establishing what has been good, and what has been bad, in the current system.

Before the OBE curriculum is introduced at tertiary level, it would be important for the current tertiary education system to objectively evaluate itself in doing the following:

i) Understand the kind of knowledge that is being transferred to students;
ii) Justify how lecturers have been awarding marks to learners; and to
iii) Assess whether the methods of assessment in Mathematics have achieved their intended objectives of separating successful learners from unsuccessful ones.

It is anticipated that the information and lessons from this exercise will improve the quality of the debate concerning the manner in which future students of the OBE curriculum should be assessed at the University level as it concerns Calculus.

The result of this research will be made available to the general public so that the Universities can engage in a meaningful discussion on how they will be able to assess student achievement outcomes within OBE and how they ought to position themselves generally for the introduction of a new OBE Mathematics curriculum at tertiary level.

1.7 Assumption of the study

We make the assumption that time demands during the Calculus examinations tend to be related to how well a student prepares for the examination. The more time a student spent revising the work and solving practice questions, the quicker he/she would be able to complete all questions on the final examination paper. Unpreparedness can manifest itself in hesitation whereby a student goes some distance in solving a problem and then cancels the work he/she has done. In this dissertation we consider hesitation to be an important element in explaining differences in performance in Calculus examination.

We also make the assumption that the solution provided by the student during the examination is as a result of examination room factors as well as intervening and background factors.

1.8 Scope of the study

This study is confined to the investigation of the assessment of Calculus students at the University first year level, using MAT 112 students of the University of South Africa as a case study.

The study is benefiting from a unique opportunity of looking back at the academic year the study is based on (year cannot be stated for ethical reasons) and being able to critique how well the students were assessed in the final Calculus examination of that year.

1.9 Definition of terms

Calculus is a branch of Mathematics that deals with limits, functions, derivatives, integrals and infinite series. This subject constitutes a major part of modern Mathematics education. It has two major branches, differential and integral Calculus, which are related by the fundamental theorem of Calculus. Calculus is the study of change, in the same way that geometry is the study of shape and algebra is the study of operations and their application to solving equations (Wikipedia/Calculus).

Problem solving in Mathematics involves getting a problem, understanding it by identifying which quantity the problem is asking one to find or solve; then devising a plan by identifying which skills and techniques one has that can be applied to solve the problem; then carry out the plan; and after getting a solution, looking back to see if the answer one gave, seems reasonable. Polya (1988) first suggested this four-step process for Mathematics problem solving.

Skill is the learned capacity to carry out a task with the minimum outlay of time, energy or both. The skill needed in Mathematics problem solving is the ability to perform each of Polya’s four steps.

1.10 Description of structure of dissertation

In Chapter One we introduced the problem and state the research question; in Chapter Two we examine the literature. In Chapter Three we describe the theoretical framework; in Chapter Four we present the methodology while in Chapter Five we present the results of the study; and in Chapter Six we give a brief description of the major findings, discuss the significance of the overall study and suggest some recommendations.

CHAPTER II: REVIEW OF RELATED LITERATURE

This chapter reviews the relevant literatures under the following sub-topics: Outcome-based education curriculum 2005, problem of poor performance of students in Mathematics, goals for understanding the first year Calculus, rigour and proof in Mathematics, human intellectual development and problem solving, Mathematics problem solving at tertiary level, contributions from cognitive Science to Mathematics problem solving, experts and novices' approaches to Mathematics problem solving, relevance of such approaches for teaching as well as learning in Open Distance Learning (ODL) environment and error analysis.

2.1 Outcome-based education Curriculum 2005

In 1997 the Council of Education Ministers (CEM) made the decision to replace old apartheid curriculum with a new Outcomes-Based Education (OBE) curriculum in the General and Further Education and Training bands. The new curriculum was introduced in General Education and Training (GET) in 1998. A new certificate, a Further Education and Training Certificate (FETC) replacing the Senior Certificate would was to be awarded for the first time in 2005. Because of this the new curriculum became popularly known as Curriculum 2005 (C2005).

C2005, the South African version of outcomes based education (OBE), is to usher in an exciting new era in South African education in which all children are promised high-quality education that will fully prepare them for life. The very term 'outcomes based education' suggests purposeful, goal directed education which, the pronouncers claim, avoids meaningless rote learning and will meet praiseworthy ideals such as the protection and enhancement of individual freedom and the development of critical thought and scientific literacy.

In Curriculum 2005, which later became National Curriculum Statement (NCS), rigid division between theory and practice; knowledge and skills; academic and applied knowledge; and between education and training would be transcended. The Mathematics component within this education policy, called Mathematical Literacy, Mathematics and Mathematical Sciences (MLMMS) specified Outcome 2 and Outcome 10, which emphasized the manipulation of number patterns and logical processes to reason correctly. Specifically, Outcome 2 and Outcome 10 stated as follows:

Outcome 2: Manipulate number patterns in different ways.

Mathematics involves observing, representing and investigating patterns in social and physical phenomena and within mathematical relationships. Learners have a natural interest in investigating relationships and making connections between phenomena. Mathematics offers ways of thinking, structuring, organizing and making sense of the world (Department of Education, 1997).

Outcome 10: Use various logical processes to formulate, test and justify conjunctures.

Reasoning is fundamental to Mathematics activity. Active learners question, conjuncture and experiment. Mathematics programs should provide opportunities for learners to develop and employ their reasoning skills. Learners need varied experiences to construct convincing arguments in problem settings and to evaluate the arguments of others (Department of Education (1997)).

After the introduction of the New Curriculum Statement (NCS), MLMMS was changed to "Mathematics" and the new statement defined the Mathematics learning area as follows:

> *Mathematics involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects. Mathematical symbols and notation form a specialized language. Five Learning Outcomes (FLO) focus on number and operations, patterns (into algebra), space and shape (geometry), measurement and data* (Department of Education, 2006)

It would appear, therefore, that the inception of the new outcome-based education Curriculum 2005 in South Africa was partly motivated by the need to create new ways of learning Mathematics effectively without resort to too much rigour but without losing the essentials of mathematical discovery.

Outcome based learning and teaching starts by defining the “outcome” of a task, and then works towards achieving it. The outcome based system in South Africa was readily exemplified by the South African Qualifications Authority (SAQA) Unit Standard 243841 in Calculus, which has these particulars:

Title: SAQA Unit Standard ID 243841

Find the derivatives of a range of functions and apply them to problems

Level: NQF LEVEL 3; Credits 6

A person credited with this unit standard is able to:

i) Represent mathematical functions graphically to enable analysis of the function.
ii) Use the three reciprocal functions in trigonometric problems.
iii) Use radian measure in working with trigonometric functions.
iv) Analyse limits and the continuity of algebraic functions using graphical representations and definitions in mathematical and contextual situations.

Critical Cross-field Outcomes (CCFO):

i) Identify the value of this unit standard by solving a variety of mathematical problems requiring graphs, trigonometry and differentiation;
ii) Interpret information in order to develop a corresponding mathematical model of the context;
iii) Use every day language and mathematical language and symbols to describe processes, theorems when solving mathematical problems;
iv) Apply this unit standard to Science by calculating areas of segments and arc lengths of circles using trigonometric functions; connect differentiation to rates of change (SAQA, 2011).

In recognizing that mathematical argument, proof and problem solving do not exist in isolation of the broader mathematical community and their application in the social world, SAQA (2011), in this Unit Standard, has specific outcomes and assessment criteria attached to it. Although these defining characteristics of the Unit Standard might be easy to appreciate from the hitherto traditional approach of teaching Calculus, the actual teaching of the Unit

Standard seems to require new approaches that emphasize a good understanding of Calculus. The CCFOs seem to want to ensure that the teaching of the Unit Standard is not removed from its possible utilization in solving practical everyday problems in society.

Given that outcome based education was meant to make teaching and learning easier, in the next subsection we discuss the problem of poor performances of students in Mathematics.

2.2 Problem of poor performances of students in Mathematics

Mathematics has a tarnished reputation in our society. It is accepted that Mathematics is difficult, obscure, and of interest only to “certain people”. The study of Mathematics carries with it a stigma, and people who are talented at Mathematics or profess enjoyment of it are often treated as though they are not quite normal (Adeyefa, 2012b).

Lack of competence in mastering Mathematics is considered a disability. Geary (2004) defines Mathematics disability as a deficit in the conceptual or procedural competencies needed for skilful performance. Many people suffer from Mathematics anxiety to some degree – just as anyone feels at least a little nervous when speaking to an audience as claimed by Adeyefa (2012b). Problem of poor performances of first year Calculus students could be due to a lot of reasons, which include anxiety, rigour and proofs.

2.2.1 Mathematics anxiety

Mathematics anxiety is a phenomenon that is often considered when examining students' problems in Mathematics. Mathematics anxiety can be a disabling condition, causing humiliation, resentment, and even panic. Ashcraft (2002, p.1) defines Mathematics anxiety as a feeling of tension, apprehension, or fear that interferes with performance in Mathematics.

Kargar, Tarmizi, and Baya (2010) describe Mathematics anxiety as feelings of tension and anxiety that interfere with the manipulation of mathematical problems in a wide variety of ordinary life and academic situations. Intellectual factors that affect Mathematics anxiety include learning styles, persistence, self-doubt, and dyslexia (Vitasari, Herawan, Wahab, Othman, and Sinnadurai, 2010). People who suffer from Mathematics anxiety feel that they are incapable of doing activities when they attend classes that involve Mathematics. Some Mathematics anxious people even have a fear of Mathematics, called maths-phobia.

The Hembree (1990) studies on meta-analysis of a group of 151 learners based on Mathematics anxiety, suggest that Mathematics anxiety is related to poor Mathematics performance in Mathematics achievement tests. Also Mathematics anxiety is related to negative attitudes of learners towards Mathematics. Hembree (1990) observes that Mathematics anxiety is directly related to Mathematics avoidance.

In his studies, Ashcraft (2002) suggests that highly anxious Mathematics students will avoid situations in which they have to solve mathematical problem; a trend of Mathematics avoidance will result in less competency, exposure and Mathematics practice, leaving students more anxious and mathematically unprepared. In college and university, anxious Mathematics students tend to feel negatively towards Mathematics. In fact, Ashcraft (2002) found that the correlation between Mathematics anxiety and variables such as confidence and motivation are strongly negative.

An important source of difficulties in Mathematics is rigour and proof which is the subject matter of the next subsection.

2.2.2 Rigour and proof in Mathematics

The dependency on unprecedented rigour and proofs in Mathematics particularly at university level appears to be a reason why Mathematics students tend to lose power to think for themselves (Nardi, 2011). This is predicated on the fact that Mathematics taught at many universities consists largely of theorems and their proofs, which students tend to memorise with little understanding. When students are required to prove a theorem they simply regurgitate the proof. Nardi (2011) attributes this to the teaching skills employed at universities. He argues that the system of lectures for teaching Mathematics prevalent at most universities has encouraged students to cope by memorising what they perceive as a fixed body of knowledge rather than learning to think for themselves.

As Yusof and Tall (1999) put it:

> *"The traditional methods of teaching Mathematics at university, which are intended to inculcate rigorous standards of mathematical proof, often seem to lead students into a 'deficit model' of rote-learning material to pass examinations…The knowledge*

gained may be appropriate for solving routine problems but it can fail in contexts requiring more conceptual insight."

Thus, according to these authors, rigour and proof have become a hindrance to learning Mathematics because students have found themselves getting "the product of mathematical thought" (i.e. relying on what others have already discovered) rather than engaging in "the process of mathematical thinking" such as the type that led to the discovery of some ingenious principles of Mathematics (for example we illustrate below, with an example of the discovery in Calculus of the product rule of differentiation by Leibniz).

Kashef, Ismail, Yusof (2011) are quoted as saying

"*Mathematical thinking is a dynamic process which expands our understanding with highly complex activities, such as abstracting, specializing, conjecturing, generalizing, reasoning, convincing, deducting, and inducting*".

At UNISA emphasis in first year Calculus has always been on mathematical thinking, problem solving and less on rigour and proof.

It needs to be emphasized that the standard of rigour in Mathematics was never uniform throughout the history of the subject. Bleug (2009) identifies Babylonian Mathematics as the most advanced and sophisticated pre-Greek Mathematics that did not embrace the concept of proof; and neither did it attempt deduction or explanation of the validity of results; but instead it dealt with specific problems; and the solutions were ingenious yet in most cases they were prescriptive. Wilder (1968) is quoted as saying:

"*The Babylonians had brought Mathematics to a stage where basic concepts in Greek Mathematics were ready to be born – the concept of a theorem and the concept of a proof*".

According to Kleiner (1991, p. 293) the axiomatic method is, without doubt, the single most important contribution of ancient Greece to Mathematics. The explicit recognition that Mathematics deals with abstractions and that proof by deductive reasoning offers a foundation for mathematical reasoning was, indeed, an extraordinary development. However, the insistence on strict, logical deduction, could have contributed to its eventual decline since concepts such as irrational numbers and the infinite were not amenable to principles of deduction; and thus a very rigorous period in Mathematics brought in its wake a long period in mathematical inactivity with very little attention paid to rigour (Kleiner, 1991, p.294). Even up to the time of the discovery of Calculus by Sir Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716), mathematical discovery heavily depended on individual insight and rough and ready methods and proofs. This could be demonstrated by Leibniz's discovery and "proof" of the product rule of differentiation which went as follows:

$d(xy)=(x+dx)(y+dy) - xy$

$= xy + xdy + ydy + dxdy - xy$

$= xdy + ydx$; considering that the quantity $dxdy$ is infinitely small in comparison with the rest, and can be discarded.

From the above, it can be concluded that the original "proof" by Leibniz was not rigorous at all, yet is a remarkable result whose truth has stood the test of time.

Similarly, many fundamental results (such as the power series expansion of cos z), were derived by Euler, using very insightful, but common sense methods (Kleiner, 1991, p. 295). In another example, Carl F. Gauss (1777-1855) was presented with a problem of finding the sum of the first hundred natural numbers; i.e. 1 + 2 +3 +….+100; he noted that if he arranged the same numbers from 100 to 1 he would get S=1 + 2 +3 + 4 +….+ 99 +100; and also

S=100 + 99 +98 +….+2 + 1; thus 2S = 101 + 101 +….+ 101 + 101 = 101 x 100; therefore, S =5050.

It could be said that rigour and proof were not inherent, or a necessary condition for the discovery of such excellent results in Mathematics. Indeed, the examples cited by Kleiner (1999) demonstrate that great mathematicians in the 18th and 19th centuries were able to use intuition and genius to discover mathematical results of great importance without necessarily having to supply modern-day rigorous proofs.

However, the preceding discussion on rigour and proof does not in any way suggest that these two aspects of Mathematics should be excluded from the university Mathematics curriculum. Even the discussions by the 24 experts in the study by Sofranas, DeFranco, Vinsonhaler, Gorgievski, Schroeder, Hamelin (2011), which is discussed in more detail below concentrated on what should be emphasized in the first year Calculus; however, that was at the expense of any discussion on how the Calculus should be taught, and what content it should have, at higher levels. The idea of restricting attention to the firm grounding of concepts and skills as well as to their applications during the first year Calculus course underlie an important assumption, namely, that the full understanding of Calculus from first year to fourth year at university must be calibrated on a gradient so that the next progression is based on a very firm ground of the earlier coverage of the materials. In this progression, the theory of Piaget is important in addressing the need to observe whether the learners had attained the readiness to assimilate materials at their different stages of intellectual development.

2.3 Piaget's theory of human intellectual development and problem solving

To illustrate the issue of human development and problem solving ability Piaget's theory of mental/intellectual development is often invoked.

According to Müller, Carpendale and Smith (2010), Piaget defined four stages of human intellectual development as: sensorimotor stage (from birth to approximately 2 years of age); preoperational stage (approximately 2-7 years of age); concrete operational stage (7 to 11 or 12 years of age); and the formal operational stage (adult level of thinking).

During the concrete operational stage which normally extends from 7 to 11 or 12 years of age, the child begins to use physical manipulation of objects to develop logical thought – hence "concrete operational". The child can now use logic to perform complex operations such as classification and ordering (Copeland, 1979, p. 24). Towards the end of this developmental stage, the pupil is able to solve mathematical problems requiring verbal abstractions; and this ability continues and is consolidated during the operational (adult) developmental stage.

This theory shows that a school teacher has the responsibility of ensuring that the problems set in his/her class are of the right difficulty depending on the intellectual development of the learners in the class. This implies that learners in one grade should be of almost equal age;

and the teacher should be a master of his/her subject, able and willing to develop the learners' abilities according to their intellectual development. However, in many South African black schools at the moment, the quality of Mathematics teaching is generally inadequate; and poor student performance in the elementary and junior schools has dire consequences for Mathematics problem solving at university level.

On the other hand, neo Piagetian theory has shown that Piaget's theory does not account for the fact that some individuals move from stage to stage faster than others and the nature of the stages is ambiguous. Research shows that the functioning of a person at a given age may be so variable from domain to domain (such as the understanding of social, mathematical and special concepts) that it is not possible to place a person in a single stage (Greenberg, 1987).

A close consideration of human cognitive development shows that:

> *"Self-awareness and self-regulation in the hyper-cognitive system also develop systematically with age. Specifically, with development, self-awareness of cognitive processes becomes more accurate and shifts from the external and superficial characteristics of problems to the cognitive processes involved. Moreover, self-representations involve more dimensions, which are better integrated into increasingly more complex structures and move along a concrete-to-abstract continuum so that they become increasingly more abstract and flexible. Furthermore, self-representations become more accurate in regard to the actual characteristics and abilities to which they refer. The knowledge available at each phase defines the kind of self-regulation that can be affected"* (Mora, 2007).

2.4 Cognitive Science and Problem solving approach in Mathematics

Callejo and Vila (2009, p.112) have cited Schoenfeld's work in which the latter was reported to have said that:

> "*there appears to be emerging consensus about the necessary scope of inquiries into mathematical thinking and problem solving. Although the fine detail varies, there appears to be a general agreement on the importance of these five aspects of*

cognition: the knowledge base, problem solving skills, monitoring and control, beliefs and affects, and practices”

Indeed, the quoted text puts together the five cited aspects as cognition; and therefore, in this dissertation, we spend time discussing the role of Cognitive Science and how it helps us to understand intellectual processes involved in problem solving. We do not discuss beliefs and affects but we have discussed the role of Cognitive Science in relation to acquiring and retaining a knowledge base, problem solving skills and to a lesser extent monitoring and control as well as on practices.

Cognitive Science investigates the nature and representation of knowledge, the structure and function of intelligence, and the relation of mind to brain and machine. In trying to understand the contributions of cognitive Science to Mathematics problem solving, we begin with the definition of a “problem”.

2.4.1 Definition of a problem

According to Jonassen (1997), solving a puzzle, deriving a formula for the sum of a geometric series, calculating the stopping distance of a car under specified speed and acceleration conditions or obtaining an indefinite integral are all common Mathematics and Science problems. Each of the above problems has an input such as a procedure to be followed, or a formula or a story with a formula, and each has one or more success criteria by which we can know that the problem has been rightly solved, including: a match in a puzzle or an integral that yields the function under the integral sign (integrand) upon differentiation, etc.

Jonassen (1997) further observed that a problem has two important characteristics, namely;

i) It is an unknown entity in a given situation with a current state and a goal state;

ii) Solving for the “unknown” must have some intellectual, social or cultural value.

That is, for an unknown entity to be a problem, someone must believe that it is worth finding the unknown. Finding the unknown is the process of problem solving. Thus, problem solving is *"any goal-directed sequence of cognitive operations"* (Jonassen, 1997).

These operations have two main attributes, namely;

> i) The mental representation of the situation in the world in which a problem solver constructs a mental representation (or mental model) of the problem;
>
> ii) Some activity based manipulation of the problem space (representations and/or the rules, or the pieces of the puzzle), whether the space is internal mental representation or it is its external physical representation.

In summary, we can describe a problem as a task for which:

i) The person confronting it wants or needs to find a solution;
ii) The person has no readily available procedure for finding the solution;
iii) The person must make an attempt to find a solution.

2.4.2 Mathematical Problem Solving

Problem solving is generally regarded as the most important cognitive activity in everyday and professional contexts. Most people are rewarded for solving problems; and many more are required to solve problems as part of everyday living (Kluwe, 1995).

As the emphasis has shifted from teaching problem solving to teaching via problem solving many authors have attempted to clarify what is meant by a problem-solving approach to teaching Mathematics. The focus is on teaching mathematical topics through problem-solving contexts and enquiry-oriented environments which are characterized by the teacher:

> *"helping students construct a deep understanding of mathematical ideas and processes by engaging them in doing Mathematics: creating, conjecturing, exploring, testing, and verifying"* (Lester, Masingila, Mau, Lambdin, dos Santon and Raymond 1994, p.154).

Specific characteristics of a problem solving approach include:

i) interactions between students/students and teacher/students (Kirtman, 2010)
ii) mathematical dialogue and consensus between students (Kirtman, 2010)

iii) teachers providing just enough information to establish background/intent of the problem, and students clarifying, interpreting, and attempting to construct one or more solution processes (Poncy, McCallum & Schmitt, 2010)

iv) teachers guiding, coaching, asking insightful questions and sharing in the process of solving problems (Lester et al., 1994)

A further characteristic is that a problem solving approach can be used to encourage students to make generalizations about rules and concepts, a process which is central to Mathematics (Windsor, 2010)).

The following example was developed from the discussions in Kamtowski (1977) and Kluwe (1995). Suppose we are asked to find the indefinite integral:

$$\int \sin^2 x \cos x dx.$$

The first step is to form an image or representation of the problem (Silver, 2008). After this, the problem solver starts to ask: have I seen this problem before; what method can I use? And the answer to this question would come from long term memory (LTM), which is a "repository of permanent knowledge and skills" (Silver, 2008). Some possibilities exist in pursuing the solution of this problem:

Suppose the LTM said "we could use the double angle theorems"; which will lead to $\sin^2 x = \cos^2 x - \cos 2x$ and

$$\sin x \cos x = \frac{1}{2}\sin 2x;$$

but this approach would lead to more complicated integrals such as:

$$\int (\cos^3 x - \cos 2x \cos x)dx \text{ or } \frac{1}{2}\int \sin 2x \sin x dx.$$

If, on the other hand, LTM said: "we should use the method of substitution" the answer would come immediately; that is:

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x + const.$$

Meta-cognition and simple differentiation would confirm that the indefinite integral obtained is the desired integral.

As a third possibility, suppose the solver did not have any of the foregoing intuition; then she will possibly sit in her chair and wonder what to do as nothing is coming from LTM, and so she would be unable to use her working memory either; therefore, meta-cognition might tell her that she should have no reason to like Mathematics. In this instance, meta-cognition would have been of no use and the end result of it all would be a negative attitude to Mathematics.

These are some of the possible scenarios that a Mathematics problem solver may encounter. In the above description, we used the terms: representations, long term memory, working memory, meta-cognition, and intuition/inspiration. All these aspects have a fundamental bearing on Calculus problem solving.

2.4.3 Problem solving and memory

When something about the real or abstract world has been learned, the information is kept in Long Term Memory (LTM).

> *"Long term memory contains mathematical knowledge, such as basic facts, processes, generalized problem types, heuristics, and algorithms. It contains beliefs and opinions about Mathematics, about one's self as a learner or doer of Mathematics, and other meta-cognitive knowledge. It also contains knowledge about the real world, knowledge about the quantities involved in a problem, and other knowledge that may be related to the problem setting. Information from long-term memory may be accessed and used in the working memory; or it may be accessed, held briefly in the working memory, and then placed into the external memory storage in the problem task environment"* (Silver, 2008)

Long-term memory is like a huge warehouse where all sorts of information are kept. A person solving a problem makes use of long term memory to provide him/her with bits of information necessary for the solution of the problem. Other information kept in long-term memory includes dates on which incidents happened, the names of people and where they live, the images that were captured by the senses of sight, smell, touch, taste and hearing; as well as abstract entities like numbers 1,2,3…, the formulas for the solution of problems, concepts, definitions, etc. For long term memory to be useful in storing readily retrievable information for problem solving, it sometimes stores chunks of information or schemata instead of single bits of information. When a problem solver tries to find the sum of n terms of a geometric series, for example, long term memory provides her with the definition, the image of a geometric progression as well as the solutions of geometric progressions that had been encountered before. Geometric progressions become one schema while arithmetic progressions retain theirs, and so do integration, complex variables and statistics.

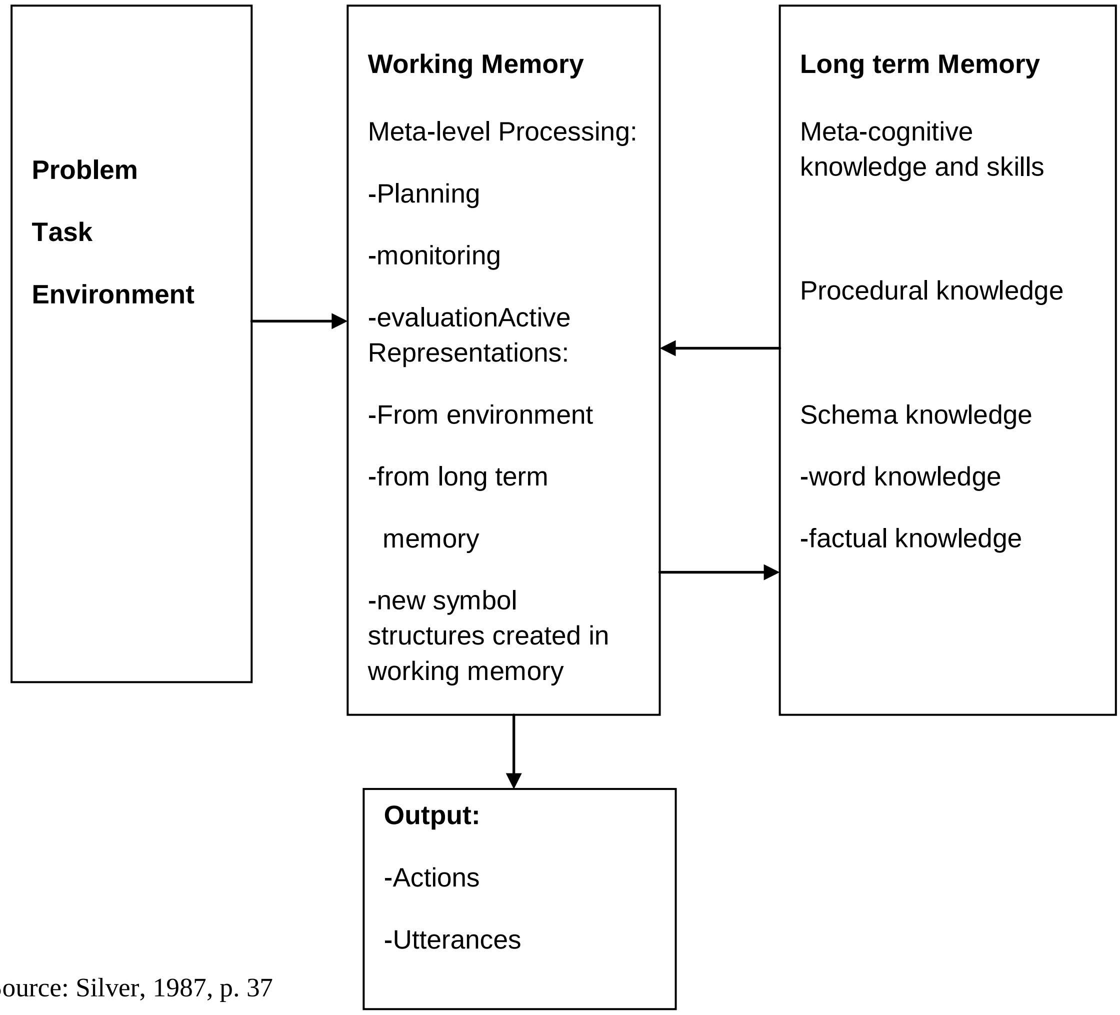
After the long-term memory had submitted a schema for solving a Calculus problem, for example, the solver uses it to perform (Calculus) calculations on other similar Calculus problems. The solver uses all these computations using short term memory, which is where immediate computational processes take place. After the calculations are completed, the short term memory hands out the output, which output is then stored in long term memory and perhaps communicated immediately to the outside world, as a solution in written form (Silver, 1987, p.43).

2.5 Cognitive Architecture

Cognitive Science has greatly borrowed from computer Science and artificial intelligence (Schoenfeld, 2011) and that is the reason why the modelled human cognitive architecture has a lot of resemblances with computer architecture. The computer architecture consists of an input unit, the central processing unit (CPU) and the hard disk. Programs and instructions get to the computer through the input unit such as a keyboard; information is processed within the CPU and data are stored on the hard disk.

Fig. 1 shows the structure of the human cognitive architecture. It shows that long term memory interacts with working memory by exchanging procedural knowledge, schema and meta-cognitive knowledge and skills either for storage or for processing. The figure also shows that information from the problem task environment goes directly to the working memory where active representations are created in order to understand the problem and then use other representations from the long term memory to process the information and find a solution. The output of the working memory goes out as actions such as typing an essay or making utterances by word of mouth. In the figure, the problem task environment represents the problem itself in which the necessary instructions and other commands are recorded.

Figure 1: The Human Cognitive Architecture



Source: Silver, 1987, p. 37

2.5.1 Meta cognition

Of all the concepts in cognitive Science, meta cognition is supreme. Meta-cognitive processes include recognizing that a problem exists, seeing the nature of the problem and selecting mental representations suitable for the task at hand (Silver, 2008). Meta-cognition also encompasses such important concepts as thinking dispositions, which embrace terms such as mindfulness. Mindfulness refers to the disposition of processing information in an open, alert and flexible way; and it also involves the investment of mental effort in learning (Silver, 2008).

The most compelling reason why meta cognition is very important, is that the concept recognizes the part played by the (human) spirit in the thinking process, and thus setting human problem solving apart from artificial intelligence; for human beings have the ability to use their minds to discern problems; they have dispositions which vary from one person to another and which cause one person to be mindful in some situations while another one might not care. The human spirit directly controls meta cognition because we are told that it determines the person’s mood, as well as the ability to monitor the progress of a solution (Davis, 1984), the disposition to continue seeking for a solution in spite of one or more failures – which is also called perseverance. A person who was unable to get the formula for the sum of the first n terms of a geometric series may give up, or, if she has the right dispositions may go on to seek for help or go to the library and work at the problem until a solution is found (and she will be the better for it, for she will have read around the subject of geometric series and probably found more interesting problems and examples that would eventually find their way into her long term memory for future reference and use).

When a problem solver attempts a particular problem, she would not always get the correct representations or the correct facts from the long-term memory. It is up to meta cognitive processes to monitor the solution and then advise on new sets of representations and other facts or schema. Problem solving then becomes a game; a game at which one plays with the hope of reaping a reward such as self-actualization and self-confidence.

2.5.2 Prior knowledge

Prior knowledge is another aspect that plays a significant role in problem solving. For one to solve a mathematical problem prior knowledge helps to develop a problem solving strategy relating the problem with what is known and then use known (i.e. prior) knowledge to determine the unknown (which is the solution).

From the literature, it seems that we all have prior knowledge as shown by the following:

> "*Even before they have received instruction in formal arithmetic, almost all children exhibit reasonably sophisticated and appropriate problem solving skills in solving simple word problems. They attend to the content of the problem; they model the problem; they invent more efficient procedures for computing the answer. Given the limits of their mathematical knowledge, this performance is remarkable"* (Carpenter, 1985).

The question is: Where did these children acquire the knowledge of Mathematics? Cognitive researchers have pointed out that new knowledge is "constructed" by the learner; that is to say, a learner does not simply add new information to their store of knowledge but instead he/she constructs new relationships among those structures. The process of building new relationships is essential to learning, as mathematical knowledge is always at least partly invented by each individual learner (Lewis, 2011).

From a pedagogical point of view, it is essential that students' prior knowledge systems be not underestimated; but instead, learners should be encouraged to drop their initial misconceptions through discussions (Barlow, Ewers, Anderson, Aragao, Baker, Boyd, Feldpausch, Gloor, Hall, Malhi, Milliken, Mulligan, Parry, Pennington, Peres, Phillips, Roman-Cuesta, Tobias, and Gardner, 2011)

According to Lovett and Greenhouse (2000) a teacher should first give a test before starting a new topic to establish what kind of misconceptions students have; and then move on to help learners avoid using their prior knowledge incorrectly. Murray, Olivier & Human (1998, p. 171) state:

> *"…leading students to discuss their view of a problem and their own tentative approaches, raises their self confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual skills".*

Thus, allowing or facilitating group work has been recommended as a way of assisting students to express their prior knowledge to their peers, and in so doing exposing their preconceived ideas.

2.5.3 Knowledge organisation

A problem solver uses working memory and long term memory to solve a problem. The processes in the working memory are temporary and this memory type has little room for lengthy details. Instead, most of the detailed knowledge is retrieved from long term memory while processes in the working memory are going on. There should, therefore, be efficient ways of organizing knowledge so that it can be easily retrieved (Schoenfeld, 2011).

One method for facilitating easy retrieval is to organize facts in a hierarchical manner. Doing this brings together pieces of related information and this is good as far as remembering facts, is concerned. Furthermore, some information is not stored individually but in chunks. When this is done routinely by the long term memory, it facilitates retrieval of data that are arranged in schemata mode, and this greatly helps in remembering (Gagne, 1978).

Thus, it is clear that the teacher should as much as possible help learners in ways that assist them to remember facts and figures arranged hierarchically. Ideas should also be integrated so that schemas are more discernible for easy retrieval by the short term memory during problem solving.

2.5.4 Problem solving: experts and novices

A problem solving expert is someone who understands a subject better and has more experience in solving problems, whereas a novice has less knowledge and skills. According to Silver (2008) a novice is not necessarily a dull person, for; we may have an expert (who is a teacher) in a classroom interacting with a group of novices who happen to be students. And the expert teacher's main mission is to assist each and every one of the students, to be an upcoming expert. The idea of being an expert does not mean that the expert knows it all;

instead, the expert uses her knowledge and skills to solve a problem and while doing so, continues to learn even more. Moreover, an expert does not necessarily always solve a problem in a shorter time than a novice and this is because the expert may take more time to understand the problem well and so be able to provide a more comprehensive solution (or more than one solution).

An expert differs from a novice in several important respects:

i) An expert has the ability to recognize patterns better than a novice. For example, when asked to find the following integrals,

$$\text{(a)} \int \frac{x^2}{(1+x)(2-x)}dx, \text{ (b)} \int \frac{x}{(1+x)(2-x)}dx, \text{ and (c)} \int \frac{1}{1+x^2}dx$$

the expert will know that problems (a) and (b) may involve the method of partial fractions while (c) may require a trigonometric substitution. This ability means that the expert does not have to call upon her working memory to store a lot of information; but instead her meta cognitive process is like an efficient storekeeper who knows where everything is stored and can go there to retrieve anything she needs, with very little effort. (In this analogy, the “store” would be the long term memory). In the literature (for instance, Silver, 2008; Fouche and Lampor, 2011), examples have been cited showing that great chess players have the ability to choose a set of moves of very high strategic value, from an almost infinite number of possible moves. This is mainly because they are able to recognise patterns with relevance in the game more easily than novice chess players can.

ii) Experts are able to retrieve information with very little effort because they have worked on many similar problems before. They have an accumulation of skills that are necessary for solving the problems. These skills are stored in long term memory where they can be recalled when necessary (Fouche and Lampor, 2011).

iii) Experts are very flexible in the way they approach a solution to a problem. This helps them to tackle a wide variety of problems. Perhaps this tendency to be flexible is due to their better meta cognition – the ability to monitor what they are doing all the time and to assess its relevance (Bransford, Brown & Cocking, 2000).

iv) When confronted with a problem, experts first think of a big picture before they immerse themselves in details. This helps them to apply their knowledge and skills in a variety of situations (Bransford, Brown & Cocking, 2000) as well as to specific situations.

2.5.5 Experts and novices : Relevance for teaching and learning

The application of knowledge relating to the differences between an expert and a novice can easily be misunderstood (Fouche and Lampor, 2011). Just because an expert is able to recognise patterns easily, does not mean than learners should be discouraged from pursuing their narrow ways of looking at a problem. Learners should be given the opportunity to develop abilities from their perspectives; some are slow learners others are fast learners. Some learners are quick to see a number of ways of doing a task, but they should not always be asked to produce a solution in ten different ways simply because they are able to do so. Some learners who use a limited number of options should be encouraged to do so, so that they can develop their skills in those areas.

However, according to Sternberg and Horvath (1995) learners should recognize the advantage of being able to see a solution from many angles; and teachers should draw the learners' attention to more possibilities than they are seeing at a particular moment. However, the teacher should realise that expertise grows slowly, and bear in mind that a learner who is able to solve a problem in one way, given another opportunity, will discover another approach. But the most valuable pedagogical lesson is for a learner to build a desire for doing something well and to continue to strive to do well in order to develop expertise; and for the teacher to develop the learner's meta cognitive abilities of: profitable mental dispositions - of wanting to learn throughout one's entire life, of openness and appreciation of new ideas, the use of evidence and logic, consideration of alternatives, creative use of the imagination, integrity and diligence. If these objectives are met, then it can be expected that a "creative novice", that is to say, the learner, will turn into an expert sooner rather than later (Fouche and Lampor, 2011).

2.6 Goals for Understanding the First Year Calculus

University of South Africa first year students who choose to go through MAT112 module are taught fundamental concepts such as derivatives, integrals and limits as well as a variety of skills to solve Calculus problems including the chain rule approach used to find the derivatives of function of a function, or the use of substitution technique to compute integrals. Students are also exposed to critical thinking areas which include the construction of relationships and connections among concepts and/or skills. Examples include, the connection between gradient and differentiation and computing derivative of a curve as the generalization of finding the gradient of a straight line. Students are also taught how to apply ideas of first year Calculus in a variety of situations so that, among other things, they could grasp the purpose of Calculus and the context in which it can be used. Such situations as finding the anti-derivatives of different types of nontrivial functions, and computing the area enclosed by two or more overlapping functions or curves, are other important topics in the Calculus curriculum.

In the study by Sofranas *et al* (2011) twenty four mathematicians with expertise in Calculus were asked the following question:

> *"What does it mean for a student to understand the first-year Calculus?"*

After they had analysed the answers to the question, the above authors concluded that there were, among other things, four overarching goals for the understanding of first-year Calculus, which are:

i) Mastery of fundamental concepts and/or skills of the first-year Calculus
ii) Construction of connections and relationships between and among concepts and skills
iii) The ability to use the ideas of the first-year Calculus; and
iv) A deep sense of the context and purpose of the Calculus

In terms of content, nearly all the experts (23 out of 24) agreed that first year Calculus students should have a mastery of the concepts and skills in dealing with derivatives as well as with integrals. Opinions were divided on limits, sequences and approximations with 70%, 37% and 33% respectively indicating that mastery of limits, sequences and approximation be the end goal for understanding the first year Calculus.

This result somehow contrasts with the situation at UNISA where limits are regarded as a precursor to the understanding of derivatives so that limits are given a considerable amount of attention in the syllabus for MAT112. Furthermore, sequences and approximation have not been part of the first year Calculus course at the University of South Africa.

Regarding the end goal of understanding the connections and relationships among concepts and skills, the study by Sofranas *et al* (2011) found that half of them agreed that students of first year Calculus need to know the fundamental theorem of Calculus, a quarter agreed to limits and half concurred on relationships between concepts and skills.

It is also noteworthy that majority of the experts (66%) regard the computation of derivatives more important than understanding of the derivative as a rate of change (50%), which in turn was more important than the graphical representation of the derivative as part of the mastery of fundamental concepts and/or skills for the first year Calculus.

As far as integration is concerned, the most important aspect identified by the study group is the mastery of techniques of integration, which was agreed to by 58% of the members. Understanding an integral as an area (25% of experts agreed) and as net change or accumulated total change was identified by 29%, which shows that these two areas of the Calculus were regarded as relatively less important by the experts.

Finally the group points out that application of derivatives and integral, non-routine problems or even problems involving models of real life phenomena are irrelevant topics in the first year Calculus course.

While analyzing the suggestions contained in Sofranas *et al* (2011) regarding the goals of the first year Calculus, it became evident that the first year Calculus course taught at UNISA contained all the major elements proposed by majority of experts participating in the study. At UNISA emphasis is placed on limits, differentiation, integration and some applications of these concepts in solving problems of differential equations.

The work by Sofranas *et al* (2011) also reported that the experts who participated in the study generally agreed that the American first year Calculus syllabus was too crowded, citing the inclusion of extraneous materials into the course which left the student to wonder what the course was all about, instead of concentrating on one or two central ideas. Extraneous materials would affect student performance and the students' ability to solve Calculus problems in examination settings. We posit that excessive rigour and proof are two such types of extraneous materials in a first year Calculus course. However, very little effort is placed on rigour and proof in the teaching of first year Calculus course at UNISA.

2.7 Problem solving approach in Calculus in an ODL environment

In South Africa, the utilitarian concept of Mathematics at tertiary level is widely recognized; students choose to do Mathematics because it leads them into different subject areas including Science, Economics, Engineering, Medicine, Management and Accountancy. Students of Calculus at the University of South Africa spend nearly nine months on the course learning and solving problems on limits, differentiation, integration and differential equations. The structure of Mathematics in general at the University of South Africa consists of axioms, determinations and theorems and to a lesser extent on their proofs, as well as problems based on applications.

The first year Calculus course at UNISA is usually taught through distance lecture method with tutorial sessions anchored mainly on problem solving. During tutorials, students work through a set of problems while the tutors are there to provide help. Some students form study groups to help each other solve mathematical problems. In distance learning, there is remote contact with the teacher; however, the student can discuss the content of her/his Calculus problems with colleagues and family members. Student centres have been established in selected regions such as Polokwane, Durban, and Bloemfontein to help students to meet one another, and for the lecturer to visit once in a while to help students with their problems. A large number of Calculus problems are also given to students so that the knowledge that has been accumulated by the student is predominantly as a result of practical understanding of solution sets to mathematical problems.

At UNISA emphasis is placed on the solution of routine problems of Calculus at first year level. The idea is to build a strong knowledge base and increase understanding of concepts and how to use them in a variety of situations. This approach resonates with the one enunciated by Carson (2007) in which he gives a critique of elements of problem solving where emphasis is on heuristics, and he goes on to stress content knowledge as opposed to teaching thinking skills devoid of content. In particular, he states that a student needs to become aware of the problem and then seek to define what the problem requires for its solution. A student who is more familiar with the problem will be better able to solve it. He summed up his position by stating that:

> "P*roblem solving would be more effective if the knowledge base and the application of that knowledge were the primary principles of the theory and practice"* (Carson, 2007, p.14).

2.8 Calculus Assessment

Assessment can be defined as a sample taken from a larger domain of content and process skills that allows one to infer student understanding of a part of the larger domain being explored. The sample may include behaviours, products, knowledge, and performances (Badders, 2010).

There is a wide range of assessment strategies that can be used in Calculus assessment which include strategies that are both traditional and alternative. The various types of alternative assessments can be used with a range of Calculus content and process skills, including the following:

i) Declarative Knowledge— the "what" knowledge
ii) Conditional Knowledge— the "why" knowledge
iii) Procedural Knowledge— the "how" knowledge
iv) Application Knowledge— the use of knowledge in both similar settings and in different contexts
v) Problem Solving— a process of using knowledge or skills to resolve an issue or problem
vi) Critical Thinking— evaluation of concepts associated with inquiry

vii) Documentation— a process of communicating understanding; and

viii) Understanding— synthesis by the learner of concepts, processes, and skills.

But the question is: How does the lecturer know that students are "getting it"? Clearly, tests that require students simply to plug numbers into formulas or recall vocabulary will not provide evidence of in-depth understanding.

Teaching, learning, and assessing mathematics in context requires careful thought and planning. In 1993, Mathematical Sciences Education Board illustrates mathematics assessment tasks that focus on thoughtful work, rather than recitation of procedures, and provides opportunities for a variety of solution strategies.

The increasing focus on the development of conceptual understanding and the ability to apply Mathematics skills is closely aligned with the emerging research on the theory of constructivism (Sofranas *et al,* 2011). This theory has significant implications for both instruction and assessment, which are considered by some to be two sides of the same coin. Constructivism is the idea that learning is an active process of building meaning for oneself. Thus, students fit new ideas into their already existing conceptual frameworks. Assessment based on constructivist theory must link the following three related issues of:

i) Student prior knowledge (and misconceptions),
ii) Student learning styles (and multiple abilities), and
iii) Teaching for depth of understanding rather than for breadth of coverage.

Meaningful assessment involves examining the learner's entire conceptual network, not just focusing on discreet facts and principles.

The main role of assessment is not to judge how well a group of students is performing as a part of educational reform, the more important use for assessment is to help the lecturer improve student learning (Popham, 2010).

Assessments are used by the lecturers to obtain information that helps them improve their instruction. This in turn helps them to bridge the gap between curriculum standards and student achievement.

Without the information gained from valid and reliable assessments, an instructional program cannot be responsive to the needs of the students. Assessment information allows the lecturer to find out whether their instruction is helping students meet criteria of mastery or make acceptable progress along academic continuums.

There are three types of assessment; formative, summative and diagnostic. Florida Center for Instructional Technology defines formative and summative assessment as follows:

Formative assessments are on-going assessments, reviews, and observations in a classroom. Teachers use formative assessment to improve instructional methods and student feedback throughout the teaching and learning process. The results of formative assessments are used to modify and validate instruction.

Summative assessments are typically used to evaluate the effectiveness of instructional programmes and services at the end of an academic year or at a pre-determined time. The goal of summative assessments is to make a judgment of student competency after an instructional phase is complete. Summative evaluations are used to determine if students have mastered specific competencies and to identify instructional areas that need additional attention.

Diagnostic Assessments can be used to direct people to the right learning experience such as a class, conversation with a Subject Matter Expert (SME), a web search, a book, an e-learning course, etc. (Badder, 2010).

Badder (2010) argues that Formative Assessments are used to strengthen memory recall by practice and to correct misconceptions and to promote confidence in one's knowledge.

Most learning environments use simple formative questions as they can:

i) Create intrigue in order to create a learning moment that motivates the learner to pay attention.
ii) Focus the students' attention towards the importance of key topics.
iii) Reduce the forgetting curve; by recalling previous knowledge and skills that will strengthen the ability to recall that knowledge or skill.

A lecturer can correct misconceptions where a student formed invalid connections; however, that does border on the purpose of a diagnostic assessment. Summative assessments, on the other hand, consist of tests and examinations designed to measure knowledge, skills, and abilities. Diagnostic assessments are not designed to strengthen memory recall; however, by their very nature they do provide some of those characteristics that do so (Badder, 2010).

A rubric is an alternative instrument used in marking/assessing. An example of a rubric, which was first developed by Szetela and Nicol (1992) was adopted and used for this study. According to Allen (2010) a scoring rubric is a scheme for classifying performance into categories that vary along a continuum. One of the strengths of a rubric is that it is criterion-referenced, rather than norm-referenced. The lecturer using the rubric will ask: "Did the student meet the criteria for level 5 of the rubric?" rather than "How well did this student do compared to other students?" This is more compatible with cooperative and collaborative learning environments than competitive grading schemes; and this is essential when using rubrics for programme assessment in situations where the instructor wants to know how well students have met his/her standards (Allen, 2010).

2.9 Solution strategies in the context of Calculus.

Developing appropriate solution strategies is an important aspect of successful mathematical problem solving. There is no hard and fast set of rules for determining the method that should be used in solving differential and integral Calculus problems. A general set of guidelines that could help identify techniques that may work is discussed below.

The following are strategies for determining the correct technique that can be used when a student is faced with an integral Calculus problem (Dawkins. 2012):

i) Simplify the integrand, if possible. This means that we should write the integrand in a form that can be dealt with, which often gives a longer and/or "messier" expression than the original integral. Many integrals can be taken from "impossible" or very difficult level to very easy level with a little simplification or manipulation. For example basic trigonometric and algebraic identities can often be used to simplify an integral. For example consider the following integral

$$\int \cos^2 x\,dx$$

This integral may not easily be evaluated as is, but by recalling the identity: $\cos^2 x = \frac{1}{2}\left(1+\cos(2x)\right)$

it becomes very easy to do.

ii) Use "simple" substitution. Always look for quick, simple substitutions before moving on to the more complicated techniques. For example consider the following integrals:

$$\int \frac{x}{x^2-1}\,dx \qquad\qquad \int x\sqrt{x^2-1}\,dx$$

The first integral can be done with partial fractions and the second could be done with a trigonometric substitution. Alternatively, both could be evaluated using the substitution $u = x^2 - 1$ and the work involved in the substitution would be significantly less than the work involved in either partial fractions or trigonometric substitution.

iii) Identify the type of integral. Any integral may fall into more than one of a given set of integral types. Because of this fact it is usually best to go all the way through the list and identify all possible types since one type may be easier than another and it is entirely possible that the easier type is placed lower in the list.

iv) Is the integrand a rational expression (i.e. is the integrand a polynomial divided by a polynomial)? If so, then partial fractions may work on the integral.

v) Is the integrand a polynomial times a trigonometric function, exponential, or logarithm? If so, then integration by parts may work.

vi) Is the integrand a product of sines and cosines, secant and tangents, or cosecants and cotangents? If so, then the techniques from trigonometric identities may work. Likewise, some quotients involving these functions can also be evaluated using these techniques.

vii) Does the integrand involve

$\sqrt{b^2 x^2 + a^2}$,

$\sqrt{b^2 x^2 - a^2}$, or

$\sqrt{a^2 - b^2 x^2}$?

If so, then a trigonometric substitution might work very well.

viii) Does the integrand have roots other than those listed above in it? If so, then the substitution

$u = \sqrt[n]{g(x)}$

might work.

ix) Does the integrand have a quadratic in it? If so, then completing the square on the quadratic might put it into a form that we can deal with.

x) Can we relate the integral to an integral we already know how to evaluate? In other words, can we use a substitution or manipulation to write the integrand into a form that does fit into the forms we have looked at previously? A typical example here is the integral:

$$\int \cos x \sqrt{1 + \sin^2 x}\, dx$$

This integral does not fit into any of the forms we have considered above. With the substitution

$$u = \sin x$$

we can reduce the integral to the form,

$$\int \sqrt{1+u^2}\, du$$

which involves a trigonometric substitution.

Similarly, the following are strategies for determining the correct technique that can be used when a student is faced with a differential Calculus problem (Dawkins, 2012). The rules of differentiation are cumulative, in the sense that the more parts a function has, the more rules that have to be applied.

Table 2: Rules of differential calculus - functions of one variable

Type of function	**Form of function**	**Graph**	**Rule**	**Interpretation**
y = constant	y = C	Horizontal line	dy/dx = 0	Slope = 0;
y = linear function	y = ax + b	Straight line	dy/dx = a	Slope = coefficient on x
y = polynomial of order 2 or higher	$y = ax^n + b$	Nonlinear, one or more turning points	$dy/dx = anx^{n-1}$	Derivative is a function, actual slope depends upon location (i.e. value of x)
y = sums or differences of 2 functions	y = f(x) + g(x)	Nonlinear	dy/dx = f'(x) + g'(x).	Take derivative of each term separately,

				then combine.
y = product of two functions,	y = [f(x) g(x)]	Typically nonlinear	dy/dx = f'g + g'f.	Start by identifying f, g, f', g'
y = quotient or ratio of two functions	y = f (x) / g (x)	Typically nonlinear	dy/dx = (f'g - g'f) / g^2.	Start by identifying f, g, f', g', and g^2
y=generalized power function	$y=\left[g\left(x\right)\right]^n$	Nonlinear	$\frac{dy}{dx}=n\left[g\left(x\right)\right]^{n-1}g'\left(x\right)$	identify g(x)
y=composite function/ chain rule	$y=f\left[g\left(x\right)\right]$ $y=f(u);\ u=g(x)$	Nonlinear	$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$	y is a function of u, and u is a function of x.
y=natural log function	$y=\ln g\left(x\right)$ $y=f\left(u\right)=\ln(u);\quad u=g(x)$	Natural log	$\frac{dy}{dx}=\frac{1}{g\left(x\right)}\cdot\frac{dg}{dx}$	Special case of chain rule
y=exponential function	$y=e^{g(x)}$ $y=f\left(u\right)=e^u;\ u=g\left(x\right)$	Exponential	$\frac{dy}{dx}=e^{g(x)}\cdot\frac{dg}{dx}$	Special case of chain rule

The following comments are important:

i) Some differential and integral Calculus problems can be done in more than one way depending on the path one takes. For example to differentiate the following function: $f(x) = \dfrac{\left(x^3 - 1\right)^4 \sqrt{3x-1}}{x^2+4}$ $\sqrt{}$

one may use quotient rule or use logarithm to simplify the function and then differentiate term by term.

ii) Some differential and integral Calculus problems may need more than one method to completely solve them. For instance a substitution may lead the student to use integration by parts or partial fractions integral.

iii) We should not get locked into the idea that an integral will only require one step to completely evaluate it. Many integrals will require more than one step.

iv) Try again. If everything that you have tried to this point does not work then go back through the process and try again. This time try a technique that you did not use the first time around.

2.10 Error analysis

Error analysis is a technique that is used to direct instructors to individualize teaching and become more sensitive to the effects of their own instruction. By becoming more sensitive to his own instruction, the teacher is assisted toward the choice of content and a suitable delivery system. Secondly, error analysis is one of a number of skills for checking whether reforms in curriculum content and in the education system have improved general pedagogy in Mathematics or not. In proper error analysis, the lecturer spends time uncovering students' mistaken ideas that led to those errors; and on the basis of this knowledge, the teacher is able to help the students replace the mistaken ideas with more accurate mathematical understanding.

In the 1970s and 1980s research studies were conducted in Mathematics education looking at students' mistakes in different mathematical domains, and Radatz (1979) has emphasized that:

> "*errors in the learning of Mathematics are the result of very complex processes. A sharp separation of the possible causes of a given error is often quite difficult because there is a close interaction among causes*".

Learning by mistakes is now regarded as the acquisition of useful "negative" knowledge; for example, detecting one's own errors helps to revise faulty knowledge structures; and storing past errors and cues that predict failure may prevent individuals from repeating mistakes (Heinze and Reiss, 2007).

Much effort has been put into trying to understand the source of misconceptions using the model of cognitive functioning when a student tries to solve a problem. Heinze (2010) argued as follows: (1) Some item(s) of information in the problem is (are) selected to act as a cue to trigger the retrieval of a seemingly appropriate schema in the cognitive structure (memory); (2) Specific information from the problem ("values") is fed into appropriate "variables" in the retrieved schema (if no values can be supplied, the schema will fill in values itself, from typical values in past experience. This is called a default evaluation); (3) Some evaluative judgment of the suitability (the "goodness of fit") of steps 1 and 2 are made (and cycling back where necessary); (4) If the judgment is that steps 1 and 2 have been successful, the result (i.e. the combination of cue information from the problem and the content of the schema) is used to continue.

Using the above framework, Heinze (2010) illustrated the solution of a quadratic equation and then looked at specific misconceptions by analyzing the way current schema mediate new learning leading to misconceptions. He remarked:

> "*One should acknowledge, of course, that errors are also a function of other variables in the education process, including the teacher, the curriculum, social factors, affective factors, emotional factors, motivation, attitudes, and possible interactions among these variables*" .

A student's ability to manage errors can be considered as a particular aspect of meta-cognition and is a prerequisite for solving complex reasoning and proof tasks (Heinze and Reiss, 2007).

2.11 Conclusion

The philosophy of Outcome-based Education is good in that Outcome based learning and teaching starts by defining the “outcome” of a task, and then works towards achieving it. Since outcome based education program assessment was introduced in 1998 it was staggered till all grades were exposed to it. As a natural progression of this system to the university, it is anticipated that university curricula will be structured along Unit Standards; therefore, the teaching of Calculus at the University of South Africa will have to be aligned to the underlying philosophy behind the development of Unit Standards, such as 243841.

Problem of poor performances of first year Calculus students could be due to a lot of reasons, which include anxiety, rigour and proofs.

However, the preceding discussion on rigour and proof does not in any way suggest that these two aspects of Mathematics should be excluded from the university Mathematics curriculum. Even the discussions by the 24 experts in the study by Sofranas *et al* (2011), discussed above concentrated on what should be emphasized in the first year Calculus; however, this was at the expense of any discussion on how the Calculus should be taught, and what content it should have, at higher levels. The idea of restricting attention to the firm grounding of concepts and skills as well as to their applications during the first year Calculus course underlie an important assumption, namely, that the full understanding of Calculus from first year to fourth year at university must be calibrated on a gradient so that the next progression is based on a very firm ground of the earlier coverage of the materials.

The implication of Piaget’s theory of human development is that learners can only solve mathematical problems that fall within their sphere of understanding. Failure to recognize this means that the teacher would set Mathematical problems that could lead to a learner being frustrated and could make the learner to hate Mathematics. Therefore, the teacher should judiciously and incrementally set problems requiring logic for learners in the concrete operational stage until they reach the adult phase of intellectual development.

It would seem in the literature that the main criticism of Piaget's theory stems from his rigid compartmentalization of stages of development. Yet, in essence, Piaget's premise of progression from one level to another is well acknowledged and is an important contribution to the theory of human intellectual development. At tertiary level, the Mathematics curriculum is made gradual from first year courses to the more advanced ones.

The role of Cognitive Science and how it helps us to understand intellectual processes involved in problem solving has been discussed in the literature. In this dissertation, we have discussed the role of Cognitive Science in relation to acquiring and retaining a knowledge base, problem solving skills and to a lesser extent monitoring and control as well as on practices. Prior knowledge is another aspect that plays a significant role in problem solving in that, for a student to solve a mathematical problem, prior knowledge helps to develop a solution strategy relevant to the problem.

The learning process is complex and educational targets are diverse, so that a wide variety of assessment tools are needed to get the appropriate information. Obtaining information about student achievement in order to guide instruction is the most important one. When used in this way, instruction improves and more closely meets the needs of the individual student, student learning is enhanced, and teachers are better able to bridge the gap between student achievement and curriculum standards (Popham, 2010). Assessment is not an end in itself. It is a means to an end, with the end being classroom decision making (Airasian, 1997). Literature discussed a variety of the use of the three types of assessment; formative, summative and diagnostic. These three types of assessments are discussed in context for evaluation of students' knowledge of Calculus.

CHAPTER III: THEORETICAL FRAMEWORK

3.1 Investigation into problem solving skills

Our main objective in this study is to determine what factors affect student performance in solving Calculus problems at first year level in an examination setting. The knowledge and skills of Calculus brought by the student to the examination room depend on a variety of factors which the student experiences, or interacts with, during the course of study.

Fayowski, Hyndman, and MacMillan (2009) credit Rosenstein (2005) with the identification of three main factors of problem solving (named Rosenstein framework), which could be grouped under three headings of content, process and personal issues.

i) Under content:

> *"the main issue is not that students do not understand concepts of Calculus, it is that they do not have skills in arithmetic and algebra"* .

In problems of differentiation, for example, students were required to make substitutions in a given function, evaluate a difference and quotient and then take limits all of which required a lot of algebra that many students at first year level do not possess. This issue is related to the background knowledge, especially of arithmetic and algebra, which students come with to the first year Calculus class.

ii) On the issue of process, teachers of Calculus ought to help students make multiple perspectives of a single concept – for example having different aspects of the notion of a function –

> *"as a rule, as an equation, as a graph, as a table or as an input-output machine"*

(and this issue was also raised by a number of experts as reported by Sofranas, DeFranco *et al*, 2011).

Still on the issue of process:

"we often ignore the grammar of Mathematics, and allow students to speak and write Mathematics incorrectly – a practice that would not be permitted in a Spanish class"

yet Mathematics is a language with its own words and symbols and rules about their use. Consequently, many students are unable to translate their answers to problems from mathematical language to the English language.

iii) On Personal issues, some students come to Calculus class thinking that they know it - they do not work until late in the year and then find it is too late to catch up.

Still on personal issues, students must learn to take responsibility for their own education (Rosenstein, 2005). At the university, students are generally working on their own and if they have not yet learned to take care of their own education, they will have a very difficult time. Finally, Fayowski *et al* (2009) emphasized that students must seek for assistance, as:

"*most students have not yet learned that it's ok to seek assistance – they have not learned that if they're having difficulties in a course, they should seek help as soon as possible"*

3.2 Essential aspects for Calculus problem solving

In this dissertation Rosenstein's framework is relevant for understanding why some students perform better than others. We also posit that other factors which have a bearing on student performance include:

i) The ability to recall facts from memory
ii) Attempts at guessing and
iii) The time pressures facing a student when answering a number of questions in a limited number of hours.

The ability to recall facts from memory is a great asset to a student of Calculus, for many results should be easily available from the student's memory to answer many Calculus problems. Students who practice hard at solving many Calculus problems develop the ability to remember many useful facts. Besides this, recourse to knowledge of arithmetic and algebra learned before joining the university depends to a large extent on a student's ability to recall the rules involved. Therefore, this dissertation has also investigated success at problem solving from the point of view of the role of cognitive Science to Mathematics problem solving.

The issue of guessing the right answer is not just limited to Mathematics alone but it extends to other study areas as well. When multiple choice questions are set on a test, the teacher is aware that one can guess at the correct answer (Burton, 2002; Prihoda et al, 2006). Attempts are therefore made to minimize the probability that a student passes the test when he is ill prepared for the examination.

According to Item Response Theory (de Ayala, 2009), the probability of a correct response to an item is a mathematical function of person and item parameters. The most important person parameter is ability or latent trait, which represents a person's intelligence or the strength of an attitude. Item parameters include difficulty, discrimination and pseudo guessing (or lower asymptote). Although, students are aware that word problems of Calculus are not in the category that would be amenable to item response theory, yet some of them resort to some form of guessing. In a large number of cases, when students have gone some distance in answering a Calculus problem (of which they know the answer but have lost their way), they often tend to force their solution toward the correct one. In this dissertation we have identified that as dishonesty.

3.3 Solving Calculus problems based on Rosenstein framework

In this dissertation we focus on solving problems of Calculus in an ODL environment under examination conditions with tight time demands. At the time of writing the examination the student would have spent a whole year learning concepts of Calculus and their applications in problem solving situations; and moreover, the student would have attempted similar assignments situations during the course of the year.

While the discussion of meta cognitive processes in Calculus problem solving was informative in helping us to understand how the student store and retrieve knowledge, there are many other factors that mediate these processes. These are called mediating factors in Rosenstein's framework (Fayowski *et al*, 2009).

Solving Calculus problems at first-year level is often associated with three main issues, which according to Rosenstein (2005) could be grouped under three headings of content, process and personal issues. Under content, he contends that many students do not have solid background in arithmetic and algebra (Fayowski *et al*, 2009)

Rosenstein (2005) further contends that the main difficulty is that in problems of differentiation, students lack tertiary level skills in Mathematics:

> *"…not only do they have to be able to carry out each of these steps individually, they also need a functioning high-level monitoring system that sees the 'big picture' that is involved in finding the derivative and that tells them what they need to do at each step".*

Most of the errors presented or reported consisted of improperly cancelling terms in fractions

> *…"we also want our students to have the appropriate facility with mathematical operations"*

and Rosenstein (2005) warns that

> *"those who are going to end up taking several semesters of Calculus in college definitely are at a disadvantage if they have difficulty with arithmetic and algebra".*

Not only do the students have to learn rules for algebraic manipulations, but also to understand the Mathematics behind those rules. Errors are often the result of misunderstanding the concepts of Mathematics, so the lecturer should spend more time uncovering the mistaken ideas that led to those errors, and helping the students replace them with more accurate mathematical understandings – through discussion of these errors in class during Group Discussion Classes, on "MyUnisa" and with each student individually who come for consultation. Lecturers should not just mark the assignments, they should also make comments.

On the issue of process as mentioned in Rosenstein's framework, the lecturer of Calculus ought to help students make connections between different topics to enable the student see the bigger picture. Multiple perspectives are important – for example enabling students to have different aspects of the notion of a function –

> *"as an equation, as a rule, as a graph, as a table, as an input-output machine – and be able to move back and forth easily among these representations".*

Lecturers should also encourage students to move back and forth between algebra and geometry; for instance in solving a system of linear equations, students should realise they are finding a point where two lines cross, when given a quadratic function, they should recognize and visualize a parabola that it defines – and be able to draw or sketch it; in other words, the equation and the graph should be seen as views of the same object. Little wonder Rosenstein conjectured that:

> *"students who can visualize algebra, who can move from algebra to geometry and back, are likely to be successful in Calculus".*

Moreover, students need to have a better sense of whether the answer they generate is reasonable. They should always be encouraged to ask the question of whether the answer they generate is reasonable or not.

Finally, on the issue of process, Rosenstein argues that:

> *"we often ignore the grammar of Mathematics, and allow students to speak and write Mathematics incorrectly – a practice that would not be permitted in Spanish class"*

yet Mathematics is a language with its own words and symbols and rules about their use. Consequently, many students are unable to translate their answers to problems from mathematical language to the English language.

On personal issues the main ideas are stated as follows:

> *"Certain students, who register for Calculus believing that they know the subject, without working hard, eventually fail the subject at the end of the session. Therefore students should be warned at the beginning of the year about this danger. Just because they know formulas for a few derivatives does not mean they know Calculus".*

Students must learn to take responsibility for their own education. University is very much unlike high school because students at high school can be cajoled into attending class. At the university, on the other hand, students work on their own, and if they have not yet learned to take care of their own education, they will have a very difficult time.

Students must ask for help and take advantage of the opportunities available to them;

> *"most students have not yet learned that it is proper to seek assistance – they have not learned that if they are having difficulties in a course, they should seek help as soon as possible"* (Rosenstein, 2005).

Although this framework is meant to address an American audience, the issues it raised have a bearing on problem solving at the tertiary level worldwide. In this dissertation, we use it as a building block in our theoretical framework.

3.4 Assessing Calculus students using the rubric

Considering the Unit Standard discussed in Chapter II, it should be noticed that the proposed method of assessment of the Unit Standard is very qualitative and could differ radically from the traditional method of assigning marks to questions on an examination. For our investigation, a rubric would be a means to providing alternative means of assessment for first year Calculus examination.

Essentially, a rubric is a scoring device that can be used for authentic assessment of student performance. It helps the teacher to determine whether a student's work meets the standard of exceptional versus that of good; it is a type of scoring guide that is used to assess more complex, subjective criteria; it is suitable for evaluation of student performance in situations that replicate the challenges of real life as opposed to isolated class tests (Rose, 2008).

Furthermore, a rubric is a device for organizing data gathered from student performance; it is able to differentiate between levels of development in a specific area of performance using scales such as: makes few/occasional/frequent mistakes in Calculus. These levels can be extended from three to six and thus facilitate sharp differentiation of performance among students and between two or more different time points. Because rubrics help set forth precise and varied criteria, teachers are better able to assess skills that may fall outside the scope of traditional testing. According to Rose (2008):

> *"Consistent scores attached to each level of a rubric, such as 1 through 4, can provide an objective basis for assigning grades"*

Some teachers find that attaching a plus or minus sign to each level, thereby creating more levels between 'beginning' and 'proficient' can help to define progress. It is also possible for a teacher to translate rubric scores into final grades, but the evidence from the literature suggests that this is generally not the main reason for using a rubric.

In this study, we develop a suitable rubric for the assessment of students' examination scripts, in a number of performance areas including:

i) Understanding of the nature of the Calculus problem in an examination setting,
ii) Development of solution strategy,
iii) Execution of the strategy through use of correct mathematical operations,
iv) Extent of hesitation in solving a problem; and
v) The extent of honesty displayed in answering a Calculus problem.

Among these performance areas, the evaluation of hesitation and the inclusion of honesty as a measurable trait were two main assessment criteria that fall outside mainstream assessment of Calculus students.

CHAPTER IV: METHODOLOGY

4.1 Research design

The investigation sets out to analyse the performance of students during the end-of-year Calculus examination (the actual year of examination is withheld to protect students' identity). It uses a retrospective study design by collecting data from a past event. A randomly chosen number of marked scripts from the examination are analysed for errors as well as for the marks awarded and on other attributes, details of which appear in the sections below.

4.2 Population, Sample and Sampling Technique

A total of 708 students' Calculus examination scripts that were collected from the Examinations Department of the University of South Africa serve as the population of the study. Out of this, 200 scripts are sampled at random, using ballot technique.

The distribution of marks (out of 100) for the 708 candidates and the distribution of the total sample of size 200 within the chosen strata were as follows:

Table 2: Selection of 200 candidates for further examination of their scripts

Performance on final exam (%)	**Number of candidates**	**Sample size selected from each performance category**	**Row Percentage**
0- 29	199	32	16.8%
30-39	116	38	32.8%
40-49	63	31	49.2%

50-69	243	80	32.9%
70 or more	87	29	33.3%
Total	708	200	28.3%

The poor performers in the 0-29 percentage range are purposively under-sampled compared to those who got 70 percent or more. This is done deliberately because the scores of the under-performers do not show much variation in the way they answered the Calculus questions as most of their attempted questions were done poorly.

4.3 Instrumentation

The development of the measuring instrument was preceded by a number of considerations. It was assumed that in the examination setting, a student carried with him the understanding and misconceptions of Calculus; he dealt with his fears and anxiety at the personal level. He was faced with a time limit and he had to answer as many questions as possible within the given time frame. He relied on his own resourcefulness when the problem was not familiar or when it was challenging; sometimes he proceeded on the right track, sometimes he met obstacles along the way; he hesitated sometimes; and at other moments he attempted to cheat - especially when he was aware of the correct answer but was not sure of the best approach.

4.3.1 The development of the assessment rubric

Ideally, the rubric should have been used in the assessment of each and every question on the examination paper but this was not possible because of the large number of questions asked. Therefore, the rubric was used as a holistic instrument, in the sense that it assessed a group of questions simultaneously. The four groups of questions were on limits, differentiation, integration and differential equations; in other words, all problems in the section on limits were assessed together; those in differentiation were treated as a unit and assessed using the same rubric, and the same was done separately for problems in integration and in differential equations.

Each of the four sections (i.e. of limits, differentiation, integration and differential equations) contained more than one problem and students solved each problem at different levels of competence; on some problems the student showed an understanding of what was needed but in others this was not so. So when it came to the application of the scale in the rubric concerning, say, “understanding what was asked”, the researcher had to rely on a general bird’s eye view of the performance of the student as far as understanding problems involving limits were concerned.

The organisation of the rubric is shown below in the Annexure as an “Analytical Scale for Problem Solving”. Group A consisted of problems on limits and students’ performance was assessed using seven scales, namely, Scale 1-7. Scale 1 was used to award a mark of 0 to 4, depending on whether the student displayed complete understanding (in which case a score of 4 was awarded) or whether the student did not understand what was given in the problem and therefore did not attempt it at all (in which case he or she got 0); or whether the student displayed some level of understanding (in which case he/she got a mark between 1 and 3). Scale 2 was on “recognizing what was needed”; Scale 3 was on developing a solution strategy; Scale 4 was on the actual answering of the problem (i.e. choosing and using correct mathematical operations); Scale 5 was on whether the student showed hesitation in answering the problem; Scale 6 was on honesty in answering the question; while Scale 7 was on “novelty” or originality or ingenuity in answering questions belonging to a section.

Similarly, Group B questions on differentiation were assessed as problems in Group A; and Group C and Group D questions were assessed similarly using the same rubric. After having assessed the four groups of questions, attention was given to assessing whether the student had displayed skill of using the knowledge of limits in differentiation as well as in integration; of using the knowledge of differentiation in differential equations; and of using knowledge of integration in solving problems in differential equations.

Altogether, the rubric gave rise to 32 quantitative variables (4 Groups of questions x 7 scales + 4) for each candidate. Data were manually collected on these 32 variables for every candidate. Because of the large amounts of data to be collected, a decision was made to limit the number of candidates studied to 200, and these 200 hundred students were then selected randomly.

In summary, the rubric required a qualitative assessment of the candidate's performance. After such qualitative assessment had been made, a mark was awarded on a scale ranging from 0 to 4. The examples below demonstrate how the rubric was scored.

On the examination paper there was the following problem on limits: Find:

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3}$$

It was assumed that in solving this problem some of the considerations would include the following: the most primitive approach would be to substitute x = -3 in the numerator and denominator and this would yield the indeterminate form $\frac{0}{0}$; after having got this far, the student should have been expected to proceed as follows:

Factorize: $x^2-9 = (x+3)(x-3)$ and factorize $x^2+2x-3 = (x+3)(x-1)$ so that

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3}$$

$$= \lim_{x \to -3} \frac{(x+3)(x-3)}{(x+3)(x-1)}$$

$$= \lim_{x \to -3} \frac{x-3}{x-1} \quad .$$

After substitution, the limit equals

$$\frac{-6}{-4} = \frac{3}{2} \, .$$

If a student had knowledge of L'Hospital's rule he would have proceeded to differentiate the numerator and denominator and then compute

$$\lim_{x \to -3} \frac{2x}{2x + 2}$$

possibly by making a direct substitution of x = -3 to get

$$\frac{-6}{-4} = \frac{3}{2};$$

and this would also be the correct solution.

In these situations the candidate would have recognized a method, which possibly he/she would have used correctly; sometimes a candidate would have made this choice but made a number of errors, say, in differentiating the numerator or denominator, or in substitution (in which case he/she would been awarded a score on the rubric less than the maximum possible mark, but commensurate with his/her effort).

4.3.2 Choice of variables

The above considerations in the solution of a limit problem gave rise to the need to choose a number of variables X1 – X8. These variables were:

X1= understanding what was given in the problem; (e.g. did the candidate see it as limit problem when x approaches infinity?). This was scored on a scale of 0 – 4.

X2 = recognizing the problem; did the candidate see it as a differential equation or as a straight forward integration problem? Did he need to draw a sketch of a graph? This was scored on a scale: 0 – 4.

X3 = did the candidate devise a good strategy e.g. by using the most efficient algorithm? This was scored on a scale: 0 – 4.

X4 = performing the necessary calculations; did the candidate follow through all the needed computations successfully? This was scored on a scale: 0 – 4.

X5 = carrying out the necessary calculations without much hesitation; if he/she scratched out an answer or made any number of cancellations, this would indicate a degree of hesitation (No hesitation = 4, a lot of cancellations = 0 on the item).

X6 = was honesty shown in the solution of the problem? In the case of the candidate who got it wrong half way, did he/she try to "bend" the solution to the kind of answer he/she believed was correct? Scale: 0 – 4

X7 = did the candidate do something novel while solving a problem? For instance, in the same limit problem above, the candidate could have reduced the quotient

$$\frac{x^2-9}{x^2+2x-3}$$

using long division to

$$1-\frac{2(x+3)}{x^2+2x-3},$$

which would have simplified to

$$1-\frac{2}{x-1}$$

whose limit as x approached –3 comes to 3/2. Such a candidate would have done something quite unusual in the solution of this limit problem and if he showed similar ingenuity on some problems of limits, he/she would have been given a 4 on the novelty scale.

X8 = did the candidate connect the knowledge required in one section to another section? Scale: 0 – 4

The following table is a summary of the rubric scales as they were applied to the four groups of questions in the final examination paper, as well as to the transfer of knowledge from one area to another.

Table 3: Rubric Scale

Group of questions	Content area	Rubric Scale (all scales were scored on 0-4)
A	Limits	Scale 1: Understanding what was given (scored 0-4) Scale 2: Recognizing what was needed (scored 0-4) Scale 3: Devising the appropriate strategy to solve the problem (0-4) Scale 4: Implementing the solution (scored 0-4) Scale 5: Indications of hesitation (reverse scale; 0=no hesitation at all while 4= much hesitation or left the problem undone) Scale 6: Honesty in answering questions (scored 0-4) Scale 7: Novelty in answering questions (scored 0-4)

B	Differentiation	Scale 1: Understanding what was given (scored 0-4) Scale 2: Recognizing what was needed Scale 3: Devising the appropriate strategy to solve the problem Scale 4: Implementing the solution Scale 5: Indications of hesitation (reverse scale; 0=no hesitation at all while 4= much hesitation or he left the problem undone) Scale 6: Honesty in answering questions Scale 7: Novelty in answering questions
C	Integration	Scale 1: Understanding what was Given (scored 0-4) Scale 2: Recognizing what was needed Scale 3: Devising the appropriate strategy to solve the problem Scale 4: Implementing the solution Scale 5: Indications of hesitation (reverse scale; 0=no hesitation at all while 4= much hesitation or he left the problem undone) Scale 6: Honesty in answering questions Scale 7: Novelty in answering questions

D	Differential equations	Scale 1: Understanding what was given (scored 0-4) Scale 2: Recognizing what was needed Scale 3: Devising the appropriate strategy to solve the problem Scale 4: Implementing the solution Scale 5: Indications of hesitation (reverse scale; 0=no hesitation at all while 4= much hesitation or he left the problem undone) Scale 6: Honesty in answering questions Scale 7: Novelty in answering questions
Transfer (AB)	A to B	One Scale - was scored on 0-4; with 0=candidate failed to use limits in differentiation, …, 4= candidate successfully transferred knowledge from limits to differentiation
Transfer (AC)	A to C	One Scale - scored on 0-4; with 0=candidate failed to use limits in integration, …, 4= candidate successfully transferred knowledge from limits to integration
Transfer (BD)	B to D	One Scale - scored on 0-4; with 0=candidate failed to use differentiation in differential equations, …, 4= candidate successfully transferred knowledge from differentiation to differential equations
Transfer (CD)	C to D	One Scale - scored on 0-4; with 0=candidate failed to use integration in differential equations, …, 4= candidate successfully transferred knowledge from integration to differential equations.

4.3.3 Validity and Reliability of the assessment rubric

The validity of the assessment rubric was determined by three experts in students' performance measurement, who were unanimous about it being appropriate for the intended purpose.

The reliability of a measuring instrument such as a rubric reflects the consistency with which results can be obtained when it is administered repeatedly. In this study, the lecturer who taught the Calculus course also collected the data using the rubric. This shows that the reliability of the rubric was a measure of the consistency of the results, if the lecturer were to score the rubric more than once. This measure of consistency is captured in a reliability coefficient that lies between 0 and 1. Ideally, for calculating the reliability of an instrument, the scoring of the rubric should be done on more than one occasion; however, techniques of splitting the measuring instrument to estimate a reliability coefficient on the basis of two halves are also available. In this study, the split-half method was used.

The split-half method is a general method used for calculating the consistency of a test when the test has been administered on one occasion only. In order to calculate the split-half reliability coefficient, the test is administered and divided into two halves, and the two halves are scored separately. Then the score of one half is compared with the score of the other half in order to find the reliability (or consistency) of the test. The Pearson's correlation coefficient, r, of the two scores is calculated; and then an adjustment is made on the value of r by the Spearman-Brown formula or Rulon's formula to produce a reliability coefficient for the entire test.

The split-half method for calculating a reliability coefficient gave one score on odd numbered items and another score on even numbered items on the rubric. In other words, the method gave rise to two scores for each assessed candidate; and then a product moment Pearson correlation coefficient was calculated for the pair of scores on the rubric (i.e. on the measuring instrument) and this correlation coefficient was used as the estimate of reliability. However, due to the theoretical limitations of this reliability measure, adjustments had to be made to the resulting Pearson's correlation coefficient to yield modified reliability estimates of the Spearman-Brown type or of the Rulon's type, which are reported for this study (Traub, 1994). Furthermore, by assuming that each item on the rubric is a test by itself, it was possible to calculate Cronbach's alpha reliability coefficient. Specifically, if $\rho^2{}_X$ and $\rho^2{}_Y$ are the respective reliability coefficients of the full questionnaire/instrument and the reliability of either half-questionnaire, then according to Spearman-Brown formula the improved reliability coefficient becomes:

$$\rho^2{}_X = \frac{2\rho^2{}_Y}{1+\rho^2{}_Y}.$$

Rulon's formula gives

$$\rho^2{}_X = 2\,[1 - \frac{(\sigma^2{}_{Y_1} + \sigma^2{}_{Y_2})}{\sigma^2{}_X}]$$

with $\sigma^2{}_{Y_1}$ and $\sigma^2{}_{Y_2}$ representing the variances of the respective total scores on the two halves while $\sigma^2{}_X$ is the variance of the score on the whole instrument. The correlation coefficient of the scores on the split halves gives an estimate of $\rho^2{}_Y$ for the full instrument.

Table 4: Validity and Reliability of the assessment rubric

Type of reliability coefficient	**Estimated Value**
Spearman-Brown split half reliability coefficient	0.987
Rulon's split half reliability coefficient	0.987
Cronbach's alpha coefficient	0.950

On the other hand, Cronbach's alpha is given by the formula:

$$\alpha = \frac{\mathrm{n}}{\mathrm{n-1}}\left(1 - \frac{\sum_{i=1}^{n}\sigma^2{}_{Y_i}}{\sigma^2{}_X}\right)$$

where n is the number of observations or scripts sampled.

Under certain conditions Cronbach's alpha gives a lower bound to the true reliability coefficient (Traub, 1994) and because of this, coefficient alpha gives a smaller value of the reliability coefficient compared to the one computed from Spearman-Brown or Rulon's formulas. On the basis of n = 200 scripts in this study, the following reliability estimates were obtained:

It can be noticed from the calculated values in Table 4 that these coefficients were very close to 1 and they show that the assessment rubric, which was constructed to assess the mathematical skills of first year Calculus students at UNISA, was an instrument of high reliability.

4.4. Data collection

Two related activities of data collection were carried out. The first activity included the collection of information on the marks that were awarded to the student on each of the four questions of the Calculus paper at the end of the academic year. The second activity dealt with gathering data using an assessment rubric that had been specifically modified from Szetela and Nicol (1992) on evaluating problem solving in Mathematics. The modified version, which was used in this study, appears in the Annex.

In addition to the information provided above concerning the scoring of the rubric, we give the following example to demonstrate other aspects of data collection.

On the UNISA Calculus paper, Question 1.b on limits, stated as follows.

Use the Squeeze or Sandwich Theorem to determine:

$$\lim_{x\to 0}\sqrt{x}\cos^2\left(\frac{1}{x}\right);\ \text{for } x > 0$$

One candidate approached the solution in the following manner:

Table 5: An example of a student's solution

Step	Student's solution	Comment
1.	$-1 \le \cos\left(\frac{1}{x}\right) \le 1$	The student was able to recognize one possible way of starting to solve the problem.
2.	$(-1)^2 \le \cos^2\left(\frac{1}{x}\right) \le (1)^2$	The left hand side of the inequality was incorrect and it led to the error in step 3 below.
3.	$1 \le \cos^2\left(\frac{1}{x}\right) \le 1$	This step was incorrect as it would imply that the middle quantity is identically equal to 1
4.	but we know that $0 \le \cos^2\left(\frac{1}{x}\right) \le 1$	The student got back on the right track
5.	$\sqrt{x}\, 0 \le \sqrt{x} \cos^2\left(\frac{1}{x}\right) \le \sqrt{x}\, (1)$	This step was executed well.
6.	where $\lim_{x\to 0} 0 \le \sqrt{x} \cos^2\left(\frac{1}{x}\right) \le 0$	The student did not use the limit symbol on the middle and right hand sides of the double inequality sign to come to the conclusion in step 7
7.	$\therefore \lim_{x\to 0} \sqrt{x} \cos^2\left(\frac{1}{x}\right) = 0$	The student got the final answer correct.

Comment on scoring the rubric

The student understood what was asked and was familiar with limit operations; he/she recognized the approach to take; he/she hatched a good strategy to solve the problem. However, the execution of the strategy was fraught with error at step 3. From the script, the candidate showed no hesitation in proceeding with the solution as no cancellations were evident; no dishonesty was displayed and no new method or technique were displayed (no novelty was exhibited). It is on the basis of these considerations that the seven scales on limits were scored. However, this scoring was based on all the problems on limits on the examination paper; therefore, the researcher had to use much personal discretion in awarding the scores on the rubric.

As another example, consider the solution of the following problem by the same candidate on integration. The solution to this problem was disastrously short and it ran as follows:

Table 6: Another example of a student's solution

Question 2.e Determine the area of the region enclosed by the curves y = cos x and $y = x^2 + 2$ on the interval $0 \le x \le 2$.

Candidate's solution

There is no area enclosed by the curves y = cos x and $y = x^2 + 2$ since $y = x^2 + 2$ has a minimum at 2 when x=0 and y = cos x has a maximum value at 1 when x=0. Therefore the curves y = cos x and $y = x^2 + 2$ do not intersect and hence we deduced that the area enclosed by the two curves y = cos x and $y = x^2 + 2$ equals zero.

Comment on scoring the rubric

Although the candidate got the solution completely wrong, he/she was aware of the shape of the two curves given. The candidate got a score of at least 1 on the rubric for knowing the shape of the two curves on the 'understanding' dimension.

4.5 Method of data analysis

The data collected for this study were analysed using mean and standard deviation, simple percentage statistics and factor analysis of the students' performance. Also other statistical tools were used to analyse the students' skills displayed.

4.5.1 Ethical issues

Proper procedures were followed to obtain from the University the Examination scripts of the students. To protect the identity of the students the year in which the examination was taken is not disclosed.

CHAPTER V: RESULTS

5.1 Performance on examination questions

The Calculus examination paper consisted of four questions; one on limits (Q1), another on differentiation (Q2) another of integration (Q3) and the last one on differential equations (Q4). Each of these questions had a number of sub-questions totalling 5 for Q1, 6 for Q2, 7 for Q3 and 6 for Q4. This means that the question paper consisted of a total of 24 different Mathematics problems to be solved within a time limit of 120 minutes (2 hours). Therefore, on average each student had just five minutes to solve one problem.

The marks on each sub-question were added together to derive a total mark for Q1 to Q4. These marks were added together to get a total, which was the indicator of performance in the Calculus examination paper. To facilitate comparisons on the relative performance on Q1-Q4, each of the four marks on the four questions was standardized to 25 for each of the students in the study.

Table 7: Performance of 200 students in the Calculus final examination

Examination Section	**Mean Mark out of 25**	**Standard deviation**
Limits: Q1	16.8	4.249
Differentiation: Q2	10.3	5.949
Integration: Q3	14.6	7.494
Differential Equations: Q4	7.3	6.900

Table 7 shows the average marks obtained on Q1-Q4. The average mark out of 25 was below 50 percent in differentiation and in differential equations. Performance in differential equations was particularly poor and the distribution of marks had a large standard deviation (s.d. = 6.900), not only because students encountered difficulties, but also because many of them did not have the time to attempt several of the sub-questions in Q4. An analysis of the sequence in which students answered questions shows that nearly all candidates started with Q1 followed by Q2 followed by Q3 and by Q4.

There was a high percentage of candidates who did not get a single mark (i.e. they got zero) on Q4. Whereas no candidate got zero (out of 25) on Q1 on limits, a staggering 29.5% of the candidates who wrote the paper got zero on Q4. This Q4 phenomenon affected the awarding of realistic performance scores using the assessment rubric.

The results showed that there was a discrepancy between candidates' performance on limits (Q1) and differentiation (Q2), yet this was not expected to be so. Table 8 shows that 69 out of 200 candidates got between 0-15 marks on Q1 (limits) but 149 candidates got the same range of marks on Q2, showing that performance on differentiation was worse than performance on limits.

Table 8: Comparing performance on Q1 and Q2 for students of Calculus

Marks out of 25 on Q1 (limits)	**Marks out of 25 on Q2 (differentiation)**		**Total number of candidates**
	0 - 15	16 - 25	
0 - 15	67	2	69
16 - 25	82	49	131
Total	149	51	200

A total of 67 candidates got 0-15 in both Q1 and Q2. And only 2 candidates who got 0-15 marks in Q1 got more than 15 on Q2. This small number (2 out of 69 candidates) is consistent with expectation since the 69 students were the weaker students; and because they had performed badly in Q1 they were not expected to do well in solving Q2 problems. However, a considerable percentage (82 out of 149 or 55%) that scored 0-15 in Q2 scored more than 15 out of 25 on Q1. This could indicate that poor performance on Q2 resulted from examination-related conditions rather than from students' abilities. Put in another way: How could so many students that had done very well in Q1 fail to do equally well on Q2? In fact, a chi-square test confirmed the presence of a discrepancy between student achievements on Q1 and Q2 (Pearson's chi-square=28.34, degree of freedom = 1, p-value<0.0001; McNemar's chi-square = 75.19, p<0.0001). Thus, there was a tendency for good students on limits to under-perform on problems in differentiation in Q2.

However, the situation was different when performance on Q1 was contrasted with performance on Q3 (on integration).

Table 9: Comparing performance on Q1 and Q3 for students of Calculus

Marks out of 25 on Q1 (limits)	**Marks out of 25 on Q3 (integration)**		**Total number of candidates**
	0 - 15	16 - 25	
0 - 15	61	8	69
16 - 25	38	93	131
Total	99	101	200

Table 9 shows that majority of students (93 out of 200) who did well on limits also performed well on integration. But a higher-than-expected number of students got low marks on questions Q1 and Q3 (61 out of 200 or 30.5%). This brought into focus some issues regarding the way the Calculus course was taught as well as how it was examined and how it was marked. These issues are discussed in the sections below.

5.2 Errors made by students

Students encountered a number of difficulties in solving problems of Calculus. As an example, we show the answer of one candidate in Figure 2 below where he was required to use the squeeze theorem to find

$$\lim_{x\to 0} \sqrt{x}\ \cos\left(\frac{1}{x}\right); x > 0.$$

The second step in the solution would only be correct if

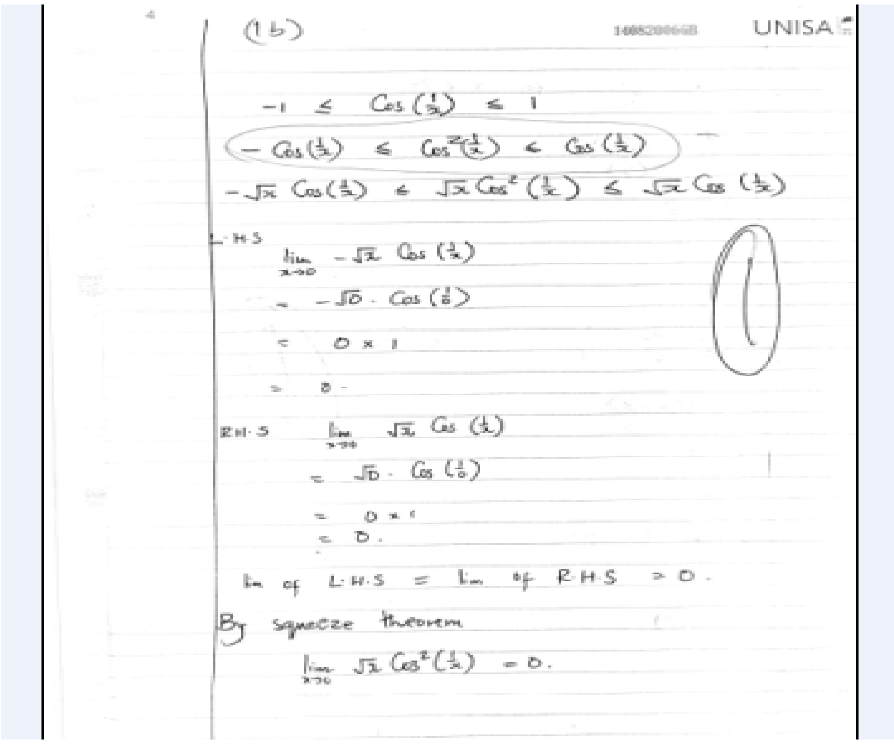
$$\cos\left(\frac{1}{x}\right) > 0$$

but this is not the case particularly when x is approaching zero and then

$$\cos\left(\frac{1}{x}\right)$$

would be oscillating between –1 and +1. The student failed to understand (or remember) the rule of inequalities which states that multiplying with a negative number changes the direction of the inequality.

Figure 1: A student solution 1



(1 b) UNISA

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$-\cos\left(\frac{1}{x}\right) \le \cos^2\left(\frac{1}{x}\right) \le \cos\left(\frac{1}{x}\right)$

$-\sqrt{x}\cos\left(\frac{1}{x}\right) \le \sqrt{x}\cos^2\left(\frac{1}{x}\right) \le \sqrt{x}\cos\left(\frac{1}{x}\right)$

L.H.S $\lim_{x\to 0} -\sqrt{x}\cos\left(\frac{1}{x}\right)$

$= -\sqrt{0}\cdot\cos\left(\frac{1}{0}\right)$

$= 0 \times 1$

$= 0$

R.H.S $\lim_{x\to 0} \sqrt{x}\cos\left(\frac{1}{x}\right)$

$= \sqrt{0}\cdot\cos\left(\frac{1}{0}\right)$

$= 0 \times 1$

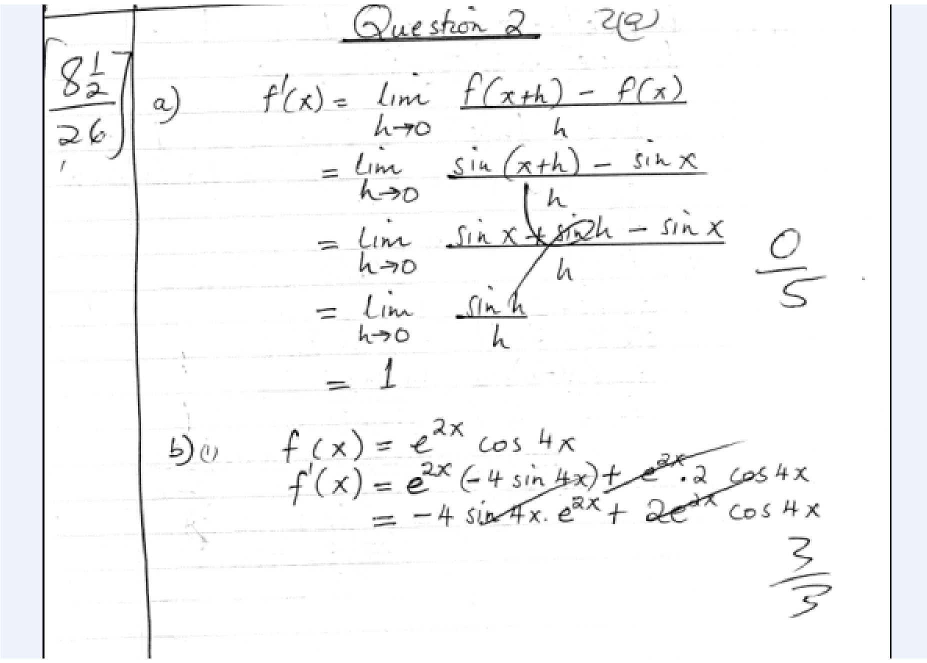
$= 0.$

lim of L.H.S = lim of R.H.S = 0.

By squeeze theorem

$\lim_{x\to 0} \sqrt{x}\cos^2\left(\frac{1}{x}\right) = 0.$

Figure 2: A student solution 2



Question 2 2(a)

$\frac{8\frac{1}{2}}{26}$

a) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0} \frac{\cancel{\sin x} + \sin h - \cancel{\sin x}}{h}$

$= \lim_{h\to 0} \frac{\sin h}{h}$

$= 1$

$\frac{0}{5}$

b)(i) $f(x) = e^{2x} \cos 4x$

$f'(x) = e^{2x}(-4 \sin 4x) + e^{2x} \cdot 2 \cos 4x$

$= -4 \sin 4x . e^{2x} + 2e^{2x} \cos 4x$

$\frac{3}{3}$

In Figure 3 the student failed to find the derivative of *f(x) = sin x* from first principles; he could not expand *sin (x+h*) and simply stated:

sin (x+h) = sin x + sin h

This showed lack of knowledge of trigonometry; yet the expansion

sin (x+h) = sin x cos h + cos x sin h

was part of the high school syllabus. The errors made in solving this problem were so serious that the candidate got zero out of 5 marks. However, his next solution to the problem of finding the derivative of

$f(x) = e^{2x} \cos 4x$

was much better. He knew the derivative of e^x and of *cos x* and he applied the product and the function-of-a-function rules correctly. However, he did not arrange his answer neatly thereby losing some marks.

The following table summarises the most common errors students made.

Table10: Summary of the most common errors

Question	Description of the error	% of sample
Q1 Limits	Failed to deal with the indeterminate form 0/0 as x approaches -3	10%
	Did not divide by x before taking the limit of f(x) as x approaches infinity	2%
	In the use of the squeeze theorem, student failed to get zero as the lower bound for cos2 (1/x).	12%
	The student failed to establish that f(x) = x2 (when x<2); = 3 (at x=2); =3x – 2 (when x>2) is not continuous at x=2	9%
Q2 Differentiation	Lack of knowledge of sum-of-angles formula for sin (x+h)	11%
	Failed to use the chain rule of differentiation for finding the derivative of f(x) = sin[ln (cos x2)].	4%
	Failed to get dy/dx of $y^2 = 3xy$ at the point (0,1) using implicit differentiation directly or by taking logarithms first.	9%
	In finding the area between two curves, the candidate sketched the graphs but failed to proceed because the curves did not intersect.	10%
	In finding the area between two curves, the candidate sketched the graphs but failed to use the given interval [0,2].	2%

Q3 Integration	The student had the idea of the fundamental theorem of Calculus but failed to differentiate using the function-of-a-function technique.	5%
	The candidate substituted correctly but could not integrate properly	8%
Q4 Differential Equations & partial derivatives	Failed to separate the variables in $dy/dx = xy/(1+x^2)$	6%
	Managed to separate the variables but failed to integrate $x/(1+x^2)$, (which could be done easily by substitution).	4%
	Attempted partial differentiation problems but had no idea about partial derivatives	5%

A large number of candidates did not attempt the last question on differential equations, which suggests that they were under time pressure to complete the paper in two hours.

5.3 Analysis of the scores from the rubric

A description of the method of data collection using the rubric was given in Section.4.4 of the last chapter. Below, we give an analysis of the scores from the rubric.

First, all the questions in the section on limits of the examination paper were assessed together using the rubric on eight dimensions, namely, how the candidate recognised the problem content, how he/she understood what was asked, the strategy that was used, the implementation of the strategy to solve the problems using correct mathematical operations, whether the candidate hesitated much or not; whether the candidate was true to himself by keeping in the chosen solution-path despite knowing that an error had been made; and whether the candidate displayed some novelty in solving problems on limits as well as whether the candidate was able to transfer some techniques of limits to differentiation.

The total score on these eight dimensions led to a single rubric mark (out of a total possible score of 32) on limits and this rubric mark was correlated with the examination mark given on the section on limits. The correlation coefficient was very highly statistically significant (*r = 0.96419, p<0.0001*).

Similar pairs of correlation coefficients were calculated for (i) exam marks on Q2 (differentiation) and rubric (total) scores on the same question on differentiation (ii) exam marks on Q3 (integration) and rubric scores on integration, and (iii) exam marks on Q4 (differential equations) and rubric scores on differential equations; and then Table 11 was obtained (see bold coefficients). All the bold correlation coefficients were significantly greater than zero at the 0.01 percent level. The high significant correlation coefficients showed that the rubric scores could as well have been used to assess students instead of the lecturer marking the examination and awarding marks in the traditional way. That means that the rubric had substitution value in that it could have been used instead.

Table 11: Correlation coefficients between Marks and Total scores

	Total scores awarded on the rubric on:			
Exam Marks awarded on:	**Limits**	**Differentiation**	**Integration**	**Differential equations**
Q1: Limits	**0.96419**	0.48981	0.47441	0.43680
Q2: Differentiation	0.45778	**0.96377**	0.71227	0.70911
Q3: Integration	0.45393	0.70675	**0.98322**	0.79486
Q4: Differential equations	0.3627	0.71127	0.75528	**0.94611**

For easy comparisons, the total rubric scores were standardized so that 25 was the maximum possible score like it was for Q1 – Q4 on the examination.

Table 12: Comparison of mean examination marks and corresponding mean rubric scores both out of 25 (standard deviations in brackets)

Examination Section	**Examination Mean Mark out of 25***	**Rubric Mean Score out of 25**
Limits	16.8 (4.249)	13.6 (3.669)
Differentiation	10.3 (5.949)	9.3 (4.377)
Integration	14.6 (7.494)	12.3 (5.356)
Differential Equations	7.3 (6.900)	7.7 (4.858)

*The examination marks appear in Table 7 above.

Table 12 shows that the patterns of marks awarded in the final examination were the same as the pattern of scores given using the assessment rubric. The highest mean was got on the section on limits followed by the mean on integration. In both assessments the differential equations section got the lowest mean. This reinforced the conclusion that the rubric was essentially a very good substitute (or possibly a good complement) for assessing students.

5.4 The superior nature of rubric assessment

It has been shown above that rubric scores tended to resemble the marks awarded to students on the four sections of the Calculus examination and therefore were a viable substitute for grading students of Calculus. However, there are a few more advantages of using an assessment rubric compared to awarding marks directly to students at the end of the examination. One of these advantages is the inclusion of qualitative-type assessments that would not be part of end-of-year examination marking, including but not limited to students' understanding of the process of solving Calculus problems, of setting about solving a problem using a feasible strategy, of assessing the extent to which students tend to hesitate in answering questions and whether the students display unethical behaviours of trying to ‘bend solutions’ toward a preconceived answer. All these could be assessed separately and given a score using a rubric.

In this study a score for ‘understanding what is required’ was calculated as a sum of the scores given to this scale on limits, differentiation, integration and differential equations. A student who had showed perfect understanding would get a total score of 16 on this dimension, so it would be possible to standardize it by normalizing to 25, so as to facilitate comparisons with earlier averages computed in the previous section. This was done for all the seven dimensions of: understanding, recognizing what was needed; laying a strategy for solving the Calculus problem; implementing the strategy by using correct mathematical operations; hesitating in implementing a strategy; honesty in answering a problem and display of ingenuity in answering Calculus problems; and Table 13 summarises the results. Also included in Table 13 was the normalized score on transferring knowledge from one area of Calculus to another.

Table 13: The status of skills and knowledge for solving first-year Calculus problems

Ability dimension	**Mean score (out of 25)**	**% Score**
Understanding Calculus problems	13.98	56
Recognizing what is required	13.96	56
Laying a solution strategy	13.24	53
Displaying honesty in solving Calculus problems	12.50	50
Implementation of a solution strategy	9.62	38
Transfer of knowledge from one area to another	8.87	35

Novelty in solving Calculus problems	6.13	25
Showing much hesitation in solving Calculus problems Showing little hesitation when solving problems	5.54 19.46	22 88

Table 13 showed that understanding Calculus problems and recognizing what was required came out well in the overall scheme of solving Calculus problems at first year level at UNISA. Average performance in these two areas was above 50 percent, and since these were averages it means that many students got more than 56% assessment on them. For both dimensions the median was 14.06 (out of 25) and the third quartile was nearly 19 (out of 25). These figures showed that majority of students understood and recognized Calculus problems.

However, Table 13 also showed that majority of students lacked originality in solving Calculus problems. The average score for novelty was 6.13 or 25 percent. Transfer of knowledge from one area to another was also poorly displayed by the students of Calculus (mean was 8.87 or 35%). That meant that majority of students saw the four areas of limits, differentiation, integration and differential equations as separate compartments. The fact that the mean score on implementation of a strategy was 9.62 (38%) showed that many students also made a lot of mistakes in moving from one step to another when solving Calculus problems. This needed to be remedied, if skill levels in Calculus are to improve. A worthwhile observation was that many students were able to lay down a clear strategy for solving Calculus problems (mean = 13.24 or 53%).

5.5 Factor Analysis

Factor analysis was used in this study because of its ability to reduce a large number of original variables and to explain these variables in terms of a few underlying common dimensions with minimum loss of information. For example, when lower secondary pupils are examined in ten subjects and it is found that Mathematics, physics and computer studies cluster together with high correlations amongst themselves, this would imply that a student who performed well in Mathematics also performed well in physics and in computer studies. Factor analysis would suggest that there is an underlying trait that is responsible for a student to perform well in the three subjects; and this trait is “mathematical ability”. Mathematical ability would be described as a factor that explained the clustering together of performance in Mathematics, physics and computer studies into a single sub-group.

Factor analysis was used in this study to analyse the observed joint patterns in the correlation matrix of the rubric scores for: understanding the problem, recognizing what was needed, laying down a solution strategy, implementing the solution strategy using correct mathematical operations, hesitating in answering questions, honesty, ingenuity or novelty, and the transfer of knowledge from one Calculus learning area to another. By studying the correlation structure, factor analysis helped us to describe how the performance in Calculus examination was influenced by a number of unobservable factors, which could possibly have policy implications in regard to the teaching and grading of Calculus content at first year level.

The method of factor extraction was a principal component analysis where the number of factors was set equal to 4. Kaiser’s Varimax rotation (Johnson & Wichern, 1994) was used and it resulted into a pattern of factor loadings shown in Table 14.

Table 14: Factor analysis of student performance in Calculus as assessed using the rubric scores

Variable	Factor Loadings For:			
	Factor 1	**Factor 2**	**Factor 3**	**Factor 4**
Understanding what was given	0.95416	-0.20014	0.20544	-0.01053
Recognizing what was wanted	0.95508	-0.20272	0.19867	-0.01690
Laying a solution	0.94469	-0.22752	0.17496	0.02125

strategy				
Implementing the strategy	0.91458	-0.17935	0.25840	0.18438
Hesitation	-0.18982	0.97748	-0.08809	-0.02665
Honesty	0.86325	-0.19387	0.36981	0.11864
Ingenuity	0.87820	-0.14478	0.24981	0.36804
Transfer of knowledge	0.75601	-0.21165	0.60715	0.07086
Variance explained by each factor	5.67548	1.22388	0.75460	0.19010
% of variance explained	70.94	15.30	9.43	2.38
Cumulative variance explained	70.94	86.24	95.67	98.05

The first factor explained 71% of total variance. All eight variables except honesty loaded onto the factor, so it could be labelled a general performance factor. The second factor, which accounted for 15.3% of total variance, is a ‘hesitation’ factor in solving Calculus problems. Factor 3 is a ‘transfer of knowledge’ factor, which reveals that we should acknowledge the role of transferring from one Calculus area to another. However, it was a minor factor, accounting for 9.4% of total variance. Factor 4 was the least important factor and it could be called 'ingenuity' factor although its correlation with novelty or ingenuity was only 0.36804. It contributed nearly 2 % of total variance.

From the factor analysis we concluded that student performance in Calculus at first year level was affected by four main factors one of which was a general factor combining understanding, recognition, and strategy, correct application of rules, honesty and ingenuity. The second factor was a hesitation factor followed in importance by transfer-of-knowledge factor and finally, by an ingenuity factor. It is worthwhile to note that the hesitation factor and the transfer-of-knowledge factors are elements that are not normally examined in mainstream Calculus assessment of students at university level. The use of an alternative assessment rubric brought these to the fore and they can be regarded as important dimensions in the successful solution of Calculus problems at tertiary level.

CHAPTER VI: DISCUSSION OF RESULTS, CONCLUSIONS AND RECOMMENDATIONS

6.1 Discussion of results

Based on the results discussed in the previous Chapter, we highlight the following discussions.

Based on the research question on the understanding of the link between limits and differentiation, this study shows that students who performed well on the first question on limits did not necessarily perform well on the next question dealing with differentiation, yet someone who has understood limits should be expected to understand differentiation from first principles. There could be several reasons for this; problems on differentiation placed heavy intellectual demands on students. This is possibly arising from the fact that students fail to transfer or integrate knowledge of limits to deal with differentiation. Transferring or integrating knowledge is an important ingredient in problem solving (Lovett and Greenhouse, 2000). It can also be argued that the knowledge of limits that students acquire and assimilate into knowledge structure of the brains, provides a foundation for future related knowledge. In this case the related knowledge happens to be knowledge of differentiation, which seems to be deficient among the participants in the current study.

In terms of difficulty, problems on limits require little efforts to find a solution. Hence majority of students were able to solve problems on limits without much difficulty. In other words, for well-prepared candidates, limit problems turned out to be routine problems whose solutions demanded easy technical knowledge and approaches (for example, the examiner asked the students to use the sandwich method of solving one limit problem, and many candidates could recall the solution approach from their long term memory). However, some problems on differentiation were difficult. The first question on differentiation asked students to use first principles to find the derivative of $\sin x$; and many students found this problem difficult, because it required the computation of

$(f(x + h) - f(x))/h$ where

$f(x) = sin(x)$

and afterwards to take the limit as *h* approaches zero. Rosenstein (2005) had observed for American students that, in general, differentiation from first principles demands much effort, as it requires making a substitution, evaluating a difference and a quotient and then taking a limit,

> *"all of which required skills in algebra which many students may not possess".*

Moreover, finding the derivative of *sin x* required knowledge of trigonometry for the expansion of sin (x+h) and to know the limit *of* $(sin\ h)/h$ as *h* approaches zero. Although the students had been introduced to similar problems at high school level, the very processes of getting to the solution were not everyday processes; and therefore, it can be argued that the solution of the problem placed heavy demands on the student to recall information that he/she had last used intensively in high school Mathematics, possibly on a "remember-this-result" basis (for example that limit of $(sin\ h)/h =1$ as h approaches zero; and moreover, high school students were not expected to know how to apply L'Hospital's rule, yet this is the rule that would yield this limit result easily). In relation to the Mathematics carried out by students from high school, Michaels *et al* (2008) had contended that "many students admitted to higher education are considerably ill-prepared for tertiary study, particularly learners from poverty-stricken and disadvantaged communities".

One could observe that some Calculus problems that were set on limits and on differentiation in the examination paper were not evenly distributed on the basis of their level of difficulty. While comparing the performance of students on the limits and differentiation sections, we should have taken the level of difficulty into consideration. The most important lesson that seems to emerge here is that the examiner should have, as much as possible aimed at setting considerably fewer questions on differentiation than on limits to try to balance the two sections on the basis of their intellectual demands on first-year Calculus students. The distribution of marks by section showed that the average marks on Q1 (limits) and Q3 (integration) were higher than on Q2 (differentiation) and on Q4 (differential equations).

On students' performance in problems on integration, a closer look at the problems on integration shows that the examiner wanted the students to use the method of substitution in solving the three problems in Q3. Moreover, two other subsequent problems on integration were amenable to the method of substitution, so it was not surprising to find that students performed well on integration. The emphasis on integration by substitution in the examination was consistent with the syllabus contents for first year Calculus. Good performance on the examination paper therefore revealed that, by doing a large number of similar problems on integration, students developed schema, which became useful in the solution of further similar problems (Silver, 1987, p. 44).

As for students' performance in problems on differential equations, there were very many students (29%) who did not get a single mark on the last question on differential equations. This was a reflection of the intense time demands on students; so much so that, by the time the student got through the first twenty-odd problems on the paper, there was no time left to solve problems on differential equations, which came at the end of the paper. In retrospect, there was a good lesson learned from this, namely, that the number of questions on the examination paper should have been balanced against the time available for the students. Failure to do this meant that working quickly through the paper became a very important, and perhaps the most important, examinable student attribute in first-year Calculus problem solving. Although the time element would always be present in any examination, yet excessive time demands should not lead to massive student failure. That would then suggest that both difficulty, and the number of questions set on the paper, should be tailored to student abilities in order to avoid massive student failure. Also, being mindful of the calibre of students was necessary to ensure that a good number of them passed the examination so that they are encouraged to develop confidence in their abilities (especially in first-year Calculus, which is a foundation for many other disciplines) toward shifting from being novices to becoming experts in Mathematics later in life (Bransford, Brown and Cocking, 2000).

On effectiveness of rubric, this study used the Analytical Scale for Problem Solving rubric adapted to Calculus problems. The scores awarded to students on the basis of the rubric correlated very well with marks awarded on the paper. This was consistent with Rose's (Rose, 2008) observation that attaching scores to each level of a rubric such as 1 through 4 can provide an objective basis for assigning grades to students. Hence, the method of assessment that uses the rubric was a potential substitute for the usual end-of-year assessment. But, as Rose (2008) had emphasized, the rubric assessment helped further in assessing skills and student attributes that fell outside the scope of traditional testing. These additional attributes include students' propensity to engage in dishonest behaviour while solving Calculus problems (for example in deliberately bending a solution-path to suit a desired or known result in Calculus).

By using the rubric scores, this study was able to assess the relative importance of understanding a Calculus problem, of recognizing what was wanted, as well as of students being able to lay a solution strategy; implementing the strategy, hesitating, and displaying ingenuity in the solution of Calculus problems. The results showed that comprehending what was given, and recognizing what was wanted were the two dimensions that came out best, both having a mean of 14 out of 25 (or 56%). Implementing a solution strategy did not come third but fourth, with a mean score of 9.62 out of 25 (or 38%). This low average score seemed to indicate that students of Calculus were not very proficient in solving problems using correct mathematical operations; in other words, many students made errors (of a mathematical nature) in proceeding from one step to another. This was likely to be a reflection of the kind of skills first-year Calculus students brought with them from high schools to university. It was therefore possible to speculate that when students came to university with poor backgrounds in Mathematics, even when the lecturer would show them the best possible solutions of selected Calculus problems, many of them failed to grasp and to apply them within a one-year time frame. Furthermore, the rubric scores also revealed that ingenuity in solving Calculus problems was at the bottom of the list; meaning that although many students did indeed solve Calculus problems correctly, yet it was only a few of them that were able to discover and apply effective shortcuts in displaying elegant and brilliant solutions.

On transfer of knowledge in the research question, the transfer of knowledge from limits to differentiation, from differentiation to integration, from differentiation to differential equations, and from integration to differential equations, was not scored very highly on the rubric at all. This suggested that more effort is needed to integrate knowledge from these different areas so that students are able to see the interconnections between them.

Many students got a good rating on the assessment rubric for answering the questions correctly. Indeed, emphasis in the course was placed on solving a large number of Calculus problems to improve the skills level of first year students. The observed good rating was consistent with Silver's observation that familiarity with problems improves meta-cognitive functions (Silver, 2008); and therefore many students were able to plan good solution strategies as a result of solving a large number of similar Calculus problems.

Factor analysis showed that problem solving skills were enhanced by general abilities including: the ability to understand what is given in the problem, recognizing what is wanted, laying out a good strategy for solving the problem, implementing the strategy well, honesty and ingenuity. These seven abilities determined the first factor, which accounted for a whopping 71 percent of total variance. Secondly, factor analysis revealed that problem solving skills were determined by a significant hesitation factor which accounted for 15.3 percent of total variance, followed by a "transfer-of-knowledge" factor, which was less important, but accounted for 9.4% of total variance. The "transfer-of-knowledge" factor could have been responsible for the observed poor performance on differentiation, which was not matched by the good performance on limits. Furthermore, an "ingenuity" factor was also extracted accounting for 2 percent of total variance (which showed that this factor was relatively insignificant). However, this result was pleasing in that ingenuity was extracted and recognized as an important component in effective mathematical problem solving at tertiary level.

Although the hesitation factor was extracted from the principal component analysis employed in this study, it was not immediately clear to us, how hesitation influenced problem solving ability; that is to say: Does it improve the score or does it reduce it? In some situations a student who first attempts a solution and then cancels it, might have got the opportunity to think through the problem better and in the end score a good mark on the question. Then the cancellation acts as a form of rehearsal that helps the student to offer a better solution strategy and its effective implementation. On the other hand, when the student is faced with intense time demands of the order of having to solve a question within an average time frame of 5 minutes (as was the case in this examination), cancellation of one's work is likely to be a disadvantage and can lead to serious loss of marks because of unfinished problems at the end of the examination paper.

In conclusion, the use of the Analytical Scale for Problem Solving in this study seems to have opened up a whole new approach to student assessment. The study found that the assessment rubric was reliable; it produced scores that correlated very well with awarded examination marks and the rubric offered possibilities of examining non-traditional dimensions such as honesty and hesitation in Calculus problem solving. Nonetheless, there is one other dimension that could have been investigated in this thesis (but it was not), concerning the ability of students to write Mathematics as a language in its own right. Rosenstein (2005) had raised the importance of teaching first year Calculus students to learn to express mathematical facts and ideas in suitable mathematical language, as indeed, it is acknowledged worldwide that good mathematical language is an important dimension of successful mathematical problem solving. However, this aspect of successful Calculus problem solving was left undone in this study and will be investigated in another research project.

6.1.1 Conceptual model for solving Calculus problems

The following major four steps are proposed for solving Calculus problems, namely; understand, plan, try it and look back.

Understand - Before a student can solve a problem he/she must first understand it. Read and re-read the problem carefully to find all the clues and determine what the question is asking.

i) What are the unknowns?
ii) What are the data?
iii) What is the condition?

Plan - Once the student understands the question and the clues, it is time to use his/her previous experience with similar problems to look for skills and tools to answer the question.

i) Does he/she know a related problem?
ii) Look at the unknown!
iii) And try to think of a familiar problem having the same or a similar unknown?

Try It - After deciding on a plan, the student should try it and see what answer he/she comes up with.

i) Write all necessary steps to the plan
ii) Check each step of the plan
iii) Can the student see clearly that the step is correct?
iv) Can the student also prove that the step is correct?

Look Back - Once the student has tried it and found an answer, he/she should go back to the problem and see if the question is really answered. Sometimes it is easy to overlook something. If the student misses something, he/she should check the plan and try the problem again.

i) Can the student check the result?
ii) Can he/she check the argument?
iii) Can he/she derive the result differently?
iv) Can he/she see it at a glance?

We apply the developed conceptual model to a question.

Find: $\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3}$

The following are the steps to follow:

Step 1. Understand.

Before a learner can solve a problem he/she must first understand that this question is a limit problem. He/she should identify the unknown as x and -3 as the data.

Step 2. Plan

The student should substitute for

$x = -3$

in the quotient problem.

When he/she obtains an indeterminate form $\frac{0}{0}$, then he/she should know that this is a cancellation type of limit problem.

The student should know how to factorise both the numerator and the denominator

Step 3. Try it.

$$\frac{x^2-9}{x^2+2x-3}=\frac{(x-3)(x+3)}{(x-1)(x+3)}$$

$$=\frac{(x-3)(x+3)}{(x-1)(x+3)}$$
$$=\frac{(x-3)}{(x-1)}$$

He/she should take the limit:

$$\lim_{x\to -3}\frac{(x-3)}{(x-1)}$$

$$=\frac{(-3-3)}{(-3-1)}$$
$$=\frac{-6}{-4}$$
$$=\frac{3}{2}$$

Step 4. Look back

The student should check that his/her substitution of $x=-3$ into the quotient problem is correct.

The student should check that his/her factorisation is correctly done.

He/she should also check that the substitution $x=-3$ into $\lim\limits_{x\to -3}\frac{(x-3)}{(x-1)}$ is correctly done.

6.1.2 Conceptual model for marking Calculus examination

The following major eight steps are proposed for marking Calculus examination questions by the module's lecturer.

Recognise - Can the student recognise the question? Can he/she find all the clues and determine what the question is asking?

Understand - Can the student understand the question? Can he/she realise the following in the question?

i) the unknowns
ii) the data
iii) the condition

Plan - Can the student use previous experience with similar problems to use appropriate tools to answer the question?

i) Does he/she know a related problem?
ii) Has he/she looked at the unknown!
iii) And has he/she tried to think of a familiar problem having the same or a similar unknown?

Try it - Does the student write all necessary steps to the plan?

i) Check each step of the plan
ii) Can the student see clearly that the step is correct?
iii) Can the student also prove that the step is correct?

Skills - Does the student demonstrate the skills to solve the problem?

i) Are correct mathematical notations used?
ii) Correct answer - Does the student answer the question? Are the following indirect questions answered?
iii) Is correct logic used?
iv) Are correct steps to solution followed?
v) Are correct mathematical operations used?

Hesitation - Does the student demonstrate hesitation in answering the question?

Cancelling and re-writing should be observed for each student.

Honesty - Does the student display honesty in solving the problem? Can the lecturer find answers to the following question?

i) Is the flow of the answer correct?
ii) Does the flow show that the student cheats or not?
iii) Does the student recover when mistakes are made?

Novelty - Does the student demonstrate novelty in answering the question?

Are the answers commensurate?

Transfer - Can the student transfer knowledge from one area to another?

The lecturer should identify if the student demonstrates different required skills in solving a problem.

We apply the developed conceptual model to a problem on integration.

Evaluate the following integral:

$$\int_0^1 \frac{t}{(t^2+1)^2}dt$$

The following are the steps to follow:

Step 1: Recognise

The student will demonstrate that he/she recognises the problem as an integral, by writing the integral symbol correctly.

Step 2: Understand

The student shows the understanding of the problem as a definite integral problem by indicating the lower and the upper limits correctly. He/she shows the understanding of the variable by indicating that he/she is integrating with respect to variable *t*.

Step 3: Plan

The student should use substitution method by changing variable as follows:

Let $u = t^2 + 1$

$$\frac{du}{dt} = 2t$$

$$dt = \frac{du}{2t}$$

Change the lower and upper limits of integration

If $t = 0$, u=1

If $t = 1$, u=2

Step 4: Try it

The student should solve the problem as follows:

$$\int_0^1 \frac{t}{(t^2+1)^2} dt = \int_1^2 \frac{t}{u^2}\frac{du}{2t}$$

$$= \frac{1}{2}\int_1^2 \frac{du}{u^2}$$

$$= \frac{1}{2}\int_1^2 u^{-2} du$$

$$= \frac{1}{2}\frac{u^{-1}}{-1}\Big|_1^2$$

$$= -\frac{1}{2(2)} - \left(-\frac{1}{2(1)}\right)$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

Step 5: Skill

The lecturer should check that correct notations are used.

Step 6: Correct answer

Lecturer should check that correct logical steps to the solution and correct mathematical operations are used in the solution, even though a correct solution is provided, as in Step 4.

Step 7: Hesitation

Lecturer should check for hesitation in the solution steps; that is, if the student writes, cancels and re-writes any step of the solution.

Step 8: Honesty

Lecturer should check honesty by considering the flow of the solution. He/she should observe if the student cheats or not. Lecturer should try to see if the student bends the solution to suit pre-conceived idea or not. The lecturer should also notice the mistakes made in the solution, if the student recovers from mistakes when they are made.

Step 9: Novelty

Lecturer should check if the student demonstrates originality in answering the question.

Step 10: Transfer of knowledge

Lecturer should check if the student demonstrates the ability to transfer the manipulation of algebra knowledge in evaluating the definite integral. For example the use of substitution and arithmetic operations in the following:

$$= \frac{1}{2}\frac{u^{-1}}{-1}\bigg|_1^2$$

$$= -\frac{1}{2(2)} - \left(-\frac{1}{2(1)}\right)$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

In summary:

This investigation is among the first of its kind to use a retrospective design to study the performance of Calculus students at the University of South Africa. The investigation has employed an assessment rubric. Its result has shown that this alternative method of assessment has far reaching potential benefits.

This study has investigated the performance of University of South Africa students in the MAT112 Calculus course, utilizing a retrospective study design. A sample of 200 scripts was selected and the performance in the four questions on the paper was analysed based on different concepts in Calculus. Each of the concepts is allocated 25 marks. The average mark on *limits* was 16.8; on *differentiation* it was 10.3; on *integration* 14.6; and on *differential equations* 7.3. Many students did not answer all the questions especially the last question on *differential equations* which came last on the question paper. The low average mark on *differential equations* can be attributed to this.

The study also utilised an assessment rubric that captured qualitative-type dimensions of students' performance. The rubric had the following 8 scales: insight into the problem, recognizing what is needed, laying a solution strategy, implementing the strategy, hesitation, honesty, novelty/ingenuity and transfer of knowledge. Data on each of the 8 scales was captured for all 200 scripts sampled.

The rubric had a split-half reliability coefficient of 0.987 and a Cronbach's alpha coefficient of 0.950. Each of the four questions on the examination paper was scored using the rubric scales and the resulting assessment gave very high positive correlations with marks awarded. This suggests that the rubric is a good alternative assessment instrument. A factor analysis of the correlation matrix of rubric scores showed that performance in Calculus depends on a general factor that accounts for 71% of total variance of rubric scores; and a "hesitation factor" and "transfer-of-knowledge factor" which accounted for 15% and 9% of total variance, respectively. The "ingenuity factor" was least important accounting for 2%. It is observed that hesitation and transfer of knowledge are not normally examined using the traditional marking systems, yet the rubric assessment method showed that these two are important dimensions affecting student performance in Calculus.

The study revealed that many topics in Calculus are covered and examined at first year level. This implied that the first year course met majority of its objectives of equipping students with the necessary skills needed for their further study of Mathematics, economics, engineering, commerce and statistics.

The study also showed that the duration of the examination was rather short, which had some negative consequences on the success of students at problem solving in Calculus. Evidence from the study shows that far too many questions were put on the paper and this resulted into some students failing to display their problem solving skills, especially concerning the questions in differential equations, which came at the end of the examination paper and which many students did not attempt.

When an assessment rubric was implemented, it demonstrated that the majority of students were capable of understanding and recognizing the nature of Calculus problems presented to them. Students were also able to display good solution skills but many of them could not successfully implement those skills.

Failure to transfer knowledge from one area of Calculus to another is another aspect, which portrayed lack of sound problem solving skills among first year Calculus students. Furthermore, the study showed that the ingenuity dimension in problem solving is very limited, and restricted to a few isolated cases, which means that the majority of students are contented with using routine methods for the solution of Calculus problems. This could, in turn, have been responsible for the observed inability of many students to transfer knowledge from one area of Calculus to another.

The study used factor analysis, which revealed that successful problem solving depends largely on a combination of understanding, recognition, and strategy. Factor analysis also revealed the presence of a hesitation factor, a transfer-of-knowledge factor and a small and insignificant ingenuity factor in successful Calculus problem solving at first year level.

6.2 Conclusion

This study has brought into focus the need to help first year Calculus students to be good problem solvers and to improve the marking skills of the module by the lecturer. The study began with discussions on the importance of Mathematics in Science and Commerce as well as in everyday activities. Literature survey was conducted concerning mathematics problem solving, mathematics discovery and the connection between human intellectual development and problem solving. Problem solving skills, the development of problem solving skills and their relevance to teaching and learning were also surveyed. The literature review ended by looking at solving Calculus problems and highlighting the fact that students need to take responsibility for their education and ought to ask for help and take advantage of opportunities available to them. The study re-assessed the work done in an end-of-year Calculus examination, both by looking at the distribution of marks awarded and by assigning new scores based on a rubric adapted for the problem at hand. Data collection was done by using a rubric and the analysis of these data shows that the rubric is a reliable instrument given that it produced scores that were very closely related to those from the final examination. The data from the rubric assisted in the assessment of dimensions that are important for problem solving in Calculus at the first year level. This study ended with the development of two conceptual models, namely;

i) A conceptual model for solving Calculus problems.

ii) A conceptual model for assessment of Calculus examination.

The former is meant for students; to guide students in solving Calculus problems, while the latter is meant for lecturer of the module; to assist Calculus lecturer in assessment of Calculus examination.

From the above analysis, this study has done much to address the objectives of the study. The main objective of the study was to:

i) Analyse the level of problem solving skills displayed by students in answering questions in a first year Calculus examination at UNISA; this was done by looking at the distribution of marks in the end-of-year examinations on different topics of Calculus as well as by looking at the rubric scores. In general students performed less well on differentiation and differential equations than on limits and integration.

ii) Analyse the type of difficulties students of Calculus are faced with in answering questions. This was done especially by looking at the kind of errors that were made while answering Calculus problems in an examination setting;

iii) Investigate alternative ways of assessing the performance of students in a Calculus examination. This was done by the use a rubric that was specially adapted for the problem at hand;

iv) Using the investigation into alternative ways of assessing the performance of students in the Calculus examination, design some possible conceptual models. This was done by coming up with two conceptual models that will improve the students' problem solving skills and alleviate the problems encountered by students on one hand, and help Mathematics teachers in marking Calculus examinations on the other.

6.3 Recommendations

Following from the observations above, the study recommends as follows:

Every effort should be directed at effectively teaching Calculus at first year level and emphasizing a wider coverage. But balance should be maintained so that students who do well on solving limit problems should also do well on differentiation. Many assignments and tests should be administered to students to reduce the hesitation factor in problem solving.

Much effort should be put in revealing to the students and encouraging them to see the interconnectedness of the various parts of the Calculus curriculum.

The study recommends that students should be helped to perform well on differential equations; many problems should be set in assignments to assist students overcome the hesitation factor; and students should be assisted to see better how knowledge in one topic (such as limits) is used in another area (such as differentiation).

6.4 Further development

The investigation has employed an assessment rubric in a limited fashion and it is suggested here, that a more extensive confirmatory retrospective study be carried out on the same population of UNISA Calculus students covering other subsequent years. The wider coverage might confirm the reliability of the assessment rubric.

Furthermore, the reasons for poor performances in the sections on differentiation and on differential equations should be investigated and addressed.

ANNEXURE I: ANALYTICAL SCALE FOR PROBLEM SOLVING

On the examination paper, questions were divided into four broad categories, which were:

Group A: Questions on Limits

Group B: Questions on Differentiation

Group C: Questions on Integration

Group D: Questions on Differential Equations

The following scales applied to Group A, Group B, Group C and Group D.

Scale 1: Understanding the Problem (Insight)

4 Complete understanding of the problem

3 Misinterpreted minor part of the problem

2 Misinterpreted major part of the problem

1 Completely misinterpreted the problem

0 No attempt

Scale 2: Recognising what was asked

4 Complete recognition of the problem

3 Recognised major part of what was needed

2 Recognised minor part of what was needed

1 Completely failed to recognise the problem

0 No attempt

Scale 3: Solving the Problem (Strategy)

4 Used a plan or strategy that could lead to a correct solution with no arithmetic errors

3 Substantially correct procedures with minor omission or procedural error

2 Partially correct procedure but with major fault

1 Totally inappropriate plan

0 No attempt

Scale 4: Answering the Problem (using correct mathematical operations)

4 This was a very good solution (80 per cent and above would be awarded)

3 This was a good solution (60 to 79 percent would be awarded)

2 A number of errors were made (40 to 59 percent would be awarded)

1 Many errors were evident in the solution

0 No answer was given or wrong answer was given

Scale 5: No hesitation in answering the problem

4 No hesitation whatsoever was observed

3 The student went back once or twice

2 The student went back many times

1 The student kept cancelling his/her solution(s)

0 Hesitated so much / the student did not even begin

Scale 6: Honesty

4 All steps were logical / no cheating was evident

3 The student cheated only once while solving problems

2 On two to three occasions the student cheated while solving problems

1 On four occasions the student cheated while solving problems

0 The student kept bending solutions to suit preconceived solutions (at least 5 times)

Scale 7: Novelty (originality in answering a question)

4 The student displayed originality (at least 5 times) while answering problems

3 The student used some amount of originality (three to four times)

2 On two occasions the student showed originality

1 Only once did the student show some originality

0 The student did not show originality at all

Scale 8: Transfer of knowledge from one area to another

A To B (Limits as applied in Differentiation)

B To C (Differentiation applied to Integration)

B To D (Differentiation applied to Differential Equations)

C To D (Integration applied to Differential Equations)

4 Student is comfortable with using knowledge from one area to another

3. Demonstrated satisfactory ability to transfer knowledge from one area to another

2 Students had little ability to transfer knowledge from one area to another

1 Student could hardly use knowledge from one area to another

0 Student was completely unable to use knowledge from one area to another

ANNEXTURE II: AN EXAMINATION PAPER

UNIVERSITY EXAMINATION

MAT112-P October/November 2007

MATHEMATICS 112

Duration : 2 Hours 100 Marks

EXAMINERS :
FIRST :
SECOND :

CALCULUSCALCULUS

This paper consists of 3 pages.

Answer **ALL** the questions.

QUESTION 1

(a) Determine the following limits:

(i) $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$ (3)

(ii) $\lim\limits_{x \to \infty} \dfrac{1 + 2x - x^2}{1 - x + 2x^2}$ (3)

(iii) $\lim\limits_{x \to \frac{3}{2}} \dfrac{2x^2 - 3x}{|2x - 3|}$ (4)

(b) Use the Squeeze Theorem to determine

$$\lim_{x \to 0} \sqrt{x} \cos^2\left(\frac{1}{x}\right); \quad x > 0. \qquad (5)$$

(c) Let

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$$

Is the function f continuous at $x = 2$ or not? (5)

[20]

[TURN OVER]

QUESTION 2

(a) Using first principles of differentiation, find the derivative of $f(x) = \sin x$. (5)

(b) Find the derivatives of the following functions using the rules for differentiation:

(i) $f(x) = e^{2x} \cos 4x$ (3)

(b) $f(x) = \sin\left[\ln\left(\cos x^3\right)\right]$ (4)

(c) Given that $xe^y - 3y\sin x = 1$, find $\dfrac{dy}{dx}$ implicitly. (4)

(d) Find the equation of the tangent and normal lines to the curve

$$y^2 = 3^{xy} \text{ at the point } (0, 1).$$ (5)

(e) Determine the area of the region enclosed by the curves $y = \cos x$ and $y = x^2 + 2$ on the interval $0 \le x \le 2$.

(5)

[26]

QUESTION 3

(a) Use the Fundamental Theorem of Calculus to evaluate $\dfrac{dy}{dx}$ if

$$y = \int_{e^{x^3}}^{0} \sin^3 t \, dt.$$ (4)

(b) Use substitution if necessary, to evaluate the following integrals:

(i) $\displaystyle\int_1^3 x^2\sqrt{x^3 + 2}\,dx$ (4)

(ii) $\displaystyle\int_0^2 \frac{e^x}{1 + e^{2x}}\,dx$ (4)

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} e^{\sin\theta} \cos\theta \, d\theta$ (4)

[TURN OVER]

(c) Determine the following indefinite integrals:

(i) $\int x^2 \sec^2 x^3 dx$ (3)

(ii) $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}} dx$ (3)

(iii) $\int \frac{x}{\sqrt{x^2+1}} dx$ (3)

[25]

QUESTION 4

(a) Solve the following differential equation:

$$\frac{dy}{dx} = \frac{xy}{1+x^2}.$$ (3)

(b) Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{3x}{4y+1}, \quad y(1) = 4.$$ (5)

(c) Find all first-order partial derivatives of

$$f(x, y) = 3e^{x^2 y} - \sqrt{x-1}.$$ (4)

(d) Given that $f(x, y) = x^4 - 3x^2y^3 + 5y$, find f_{xx} ; f_{xy} ; f_{xyy}. (6)

(e) Suppose a bacterial culture initially has 400 cells. After 1 hour, the population has increased to 800. What will the population be after 10 hours? (5)

(f) Suppose that the value of a R40 000 asset decreases at a constant rate of 10%. Find its worth after:

(i) 10 years.

(ii) 20 years.

(6)

[29]

TOTAL: [100]

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